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From discrete to continuous time

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Abstract

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A general metatheorem is proved which reduces a wide class of statements about continuous time stochastic processes to statements about discrete time processes. We introduce a strong language for stochastic processes, and a concept of forcing for sequences of discrete time processes. The main theorem states that a sentence in the language is true if and only if it is forced. Although the stochastic process case is emphasized in order to motivate the results, they apply to a wider class of random variables. At the end of the paper we illustrate how the theorem can be used with three applications involving submartingales.

0. Introduction

Many theorems about stochastic processes come in two similar forms, an 'easy' discrete time result and a 'hard' continuous time result. In this paper we present a general metatheorem which reduces a wide class of statements about continuous time processes to statements about discrete time processes. The theorem takes care of much of the hard work which is often needed to get from discrete to continuous time results, and has a variety of applications. We introduce a strong language for stochastic processes, and a concept of forcing for sequences of discrete time processes. The main theorem (Theorem 5.3) states that a sentence in the language is true if and only if it is forced. The statement and proof of the theorem involves notions from Robinson's infinitesimal analysis. However, once the machinery is in place, applications of the theorem typically involve only elementary arguments about standard discrete time stochastic processes. Although the stochastic process case is emphasized in this paper to motivate the results, they apply to a wider class of random variables. At the end of the paper we illustrate how the theorem can be used with three applications involving submartingales.

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The classical counterpart of the method developed here is a weak convergence argument, where a random variable with values in a Polish space M is constructed by taking a limit point of a tight sequence of simpler random variables in the weak topology (for example, see Billingsley [6]). Often, the random variables are stochastic processes, i.e. M is a space of functions from $[0, \infty)$ into $\mathbb{R}$, and the simpler stochastic processes are random piecewise linear or step functions. These simpler processes are 'discrete time approximations' of the final process. A classical example of the method is the construction of Brownian motion as a weak limit of random walks. An easier construction of Brownian motion was given by Anderson [2] using the Loeb measure on a hyperfinite probability space, and his method has been applied in a variety of other weak convergence situations.

Weak convergence arguments may become messy for several reasons. It is often difficult to show that a sequence of processes is tight. Given a tight weakly convergent sequence of stochastic processes on one probability space, the 'limit' is usually a process on another probability space. For this reason, one obtains only weak forms of many statements in $\forall\exists$ form; instead of proving $\forall x\, \exists y\, \phi(x, y)$ one gets something like $\forall x\, \exists \bar{x}\, \exists \bar{y}\, \phi(\bar{x}, \bar{y})$ where $\bar{x}$ and $\bar{y}$ are on a different probability space and $\bar{x}$ is as much like x as possible. One is then led to various notions of processes on different probability spaces being alike. Another difficulty is that one may need to show that a process has properties which are not preserved by weak convergence, such as properties related to conditional expectations.

The nonstandard approach to the subject (see, for example, [1]), has the advantage that one always works on a fixed probability space, namely a hyperfinite probability space with a Loeb measure. Weak convergence arguments are replaced by lifting theorems, showing that a standard process has a property if and only if it has a lifting in the nonstandard universe with an analogous property. The difficulty in showing that a sequence is tight is converted into the difficulty in proving a pushing down theorem which is needed to come back down from a process in the nonstandard universe to a standard process. The method developed here may be regarded as a way of packaging a family of lifting theorems and pushing down theorems for a variety of properties into a single theorem which can be used without an excursion into the nonstandard universe.

Sections 1 and 2 contain preliminary material on Polish spaces and hyperfinite probability spaces. Section 3 contains characterizations of tight sets of random variables on a hyperfinite probability space. In Section 4 we introduce the central notion of a liftable function which maps random variables on one hyperfinite space to random variables on another. The forcing machinery and main theorem are in Section 5. A many-sorted language $\mathcal{L}$ is introduced which has universal and existential quantifiers over random variables, countable connectives, and atomic formulas which are equations between liftable functions of random variables. A notion of forcing is introduced where the conditions are infinite sets of natural numbers, and statements are formulas of $\mathcal{L}$ with names which are sequences of simple random variables. It is proved that if all the names in a statement ϕ converge on a condition p, then p forces ϕ if and only if ϕ is true. In Section 6, the expressive power of the language $\mathscr{L}$ is built up by showing that a large class of functions are liftable ($\mathscr{L}$ has greater expressive power than the adapted probability logic from [19]). Section 7 gives criteria for forcing statements involving expected values and conditional expectations.

Section 8 illustrates how the main theorem can be applied with three examples of proofs of existence theorems in probability theory. To keep the required background to a minimum, each example involves the notion of a submartingale. The strategy is to prove a theorem about continuous time stochastic processes by showing that it is forced. The proofs deal only with discrete time stochastic processes. The first application, Theorem 8.2, shows that for every pair x_t, y_t of nonnegative submartingales on a hyperfine probability space there is a martingale m_t whose values are always equal to x_t or $-y_t$. This improves earlier results along this line, which dealt with the case that $x_t = y_t$. In that case, Gilat [11] proved a weak existence theorem giving an m_t on a different probability space than x_t, and Barlow [4] and Perkins [26] proved strong existence theorems. Theorem 8.3 is a decomposition theorem for submartingales. It states that for every submartingale x such that the square of the conditional variation of x is integrable, there exists an increasing process z with values in the set of natural numbers and a martingale m such that $x = z + m$. Example 8.4 is a proof of a strong existence theorem for stochastic integral equations from [16] and [17], again on a hyperfinite probability space.

A wider variety of applications, which involve more background from probability theory, will be given in a future publication. The advantage of the method given here is that the main theorem takes care once and for all of many of the difficulties which are involved in passing from discrete to continuous time processes. Instead of going through a weak convergence argument as in the book [6], or a lifting theorem as in the book [1], one checks that appropriate statements are expressible in the language $\mathscr{L}$ and shows that they are forced.

The ideas in this paper developed from the Loeb measure construction [23], Anderson's construction of Brownian motion [2], the application of these ideas to prove existence theorems in probability theory (see [18] and [1]), and adapted probability logic which was introduced in [19] and [15]. Special thanks are due to Sergio Fajardo, Douglas Hoover, and Steve Kosciuk for many helpful conversations on the subject of this paper.

1. Preliminaries on Polish spaces

In this section we review some facts which we shall use concerning Polish spaces (i.e., complete separable metrizable topological spaces). We begin with a review of the notions of tightness and relative compactness. For more details and proofs of the results see [6] or [9].

We let $\mathbb{N}$ denote the set of positive integers. The symmetric difference between two sets A and B is denoted by $A \triangle B$. The characteristic function of a set A will be denoted by $\mathbb{1}_A$.

We assume throughout this paper that L, M and N are Polish spaces, and that ρ is a metric for M.

A subset $A \subseteq M$ is said to be *relatively compact* iff its closure is compact. A sequence $\langle y_n \rangle$ in M is *relatively compact* iff its range is relatively compact.

1.1. A sequence $\langle x_n \rangle$ in M is relatively compact if and only if every subsequence of $\langle x_n \rangle$ has a convergent subsequence.

Definition. Meas(M) denotes the space of all Borel probability measures on M with the weak topology, that is, the weakest topology such that for each bounded continuous function $\phi : M \to \mathbb{R}$, the function $\mu \mapsto \int \phi \, d\mu$ is continuous on Meas(M).

1.2. *If M is a Polish space, then* Meas(M) *is also a Polish space.*

Definition. A set of measures $T \subseteq \text{Meas}(M)$ is said to be *tight* iff for every $n \in \mathbb{N}$ there is a compact set $K_n \subseteq M$ such that for all $\mu \in T$, $\mu(K_n) \geq 1 - 1/n$.

1.3. Prohorov's Theorem. *A set $T \subseteq$* Meas(M) *is tight if and only if T is relatively compact in* Meas(M).

Definition. Let (Ω, P) be a complete probability space. We shall say that two P-measurable functions, or random variables, $x, y : \Omega \to M$ are *equivalent* iff $x(\omega) = y(\omega)$ P-almost surely. Abusing notation, the equivalence class of x is also denoted by x. Let $\text{rv}(\Omega, M)$ be the space of all equivalence classes of P-measurable random variables $x : \Omega \to M$ with the metric

$$\hat{\rho}(x, y) = \inf\{\varepsilon \in \mathbb{R} : P\{\omega \in \Omega : \rho(x(\omega), y(\omega)) \leq \varepsilon\} \geq 1 - \varepsilon\}.$$

We see from the definition that convergence in the metric $\hat{\rho}$ is the same as convergence in probability with respect to the measure P.

Following usual practice, we shall often suppress the variable ω. For instance, we may write "$x \leq y$" to abbreviate "$x(\omega) \leq y(\omega)$ for almost all $\omega \in \Omega$". Given a property $\phi(\omega)$, we may write $P[\phi]$ for $P\{\omega \in \Omega : \phi(\omega)\}$, and given an integrable function $x(\omega)$, we may write $E[x]$ for the expected value $E[x(\omega)]$.

Each random variable $x \in \text{rv}(\Omega, M)$ induces a probability measure μ on M by the rule $\mu(A) = P(x^{-1}(A))$ for each Borel set $A \subseteq M$. The measure induced by x is called the *law* of x. Thus law is a function from $\text{rv}(\Omega, M)$ into Meas(M). A set $T \subseteq \text{rv}(\Omega, M)$ of random variables is said to be *tight* if the set $\{\text{law}(x) : x \in T\}$ is tight.

In this paper we shall often work with the space $\mathrm{rv}(\Omega, M)$ where Ω is a hyperfinite set and P is a Loeb probability measure on Ω.

We now give a brief review of characterizations of closed sets, compact sets, and continuous functions on Polish spaces using Robinson's analysis. Throughout this paper we work with an ω_1-saturated nonstandard universe.

Definition. If $X, Y \in {}^*M$, $X \approx Y$ means that ${}^*\rho(X, Y) \approx 0$. X is *near-standard* iff there exists $y \in M$ such that $X \approx {}^*y$. $\mathrm{ns}({}^*M)$ is the set of near-standard $X \in {}^*M$, and for $X \in \mathrm{ns}({}^*M)$, the *standard part* of X is the unique point $x = \mathrm{st}(X) \in M$ such that $X \approx {}^*x$. For $A \subseteq {}^*M$, we define

$$\mathrm{st}(A) = \{\mathrm{st}(X) : X \in A \cap \mathrm{ns}({}^*M)\}.$$

For $B \subseteq M$, we define

$$\mathrm{st}^{-1}(B) = \{X \in \mathrm{ns}({}^*M) : \mathrm{st}(X) \in B\}.$$

1.5 (Robinson [27]). (a) *If $A \subseteq {}^*M$ is internal, then* $\mathrm{st}(A)$ *is closed.*

(b) *A set $B \subseteq M$ is closed if and only if $B = \mathrm{st}({}^*B)$.*

(c) *A set $B \subseteq M$ is relatively compact if and only if ${}^*B \subseteq \mathrm{ns}({}^*M)$.*

(d) *A set $B \subseteq M$ is compact if and only if $B = \mathrm{st}({}^*B)$ and ${}^*B \subseteq \mathrm{ns}({}^*M)$.*

(e) *If $X \in {}^*M$ and for each $n \in \mathbb{N}$ there is a near-standard Y with ${}^*\rho(X, Y) \leqslant 1/n$, then X is near-standard.*

(f) *The set* $\mathrm{ns}({}^*M)$ *and function* $\mathrm{st} : \mathrm{ns}({}^*M) \to M$ *depend only on the topology on M, not on the metric ρ.*

Note, however, that the relation $X \approx Y$ on ${}^*M$ does not depend on the metric ρ when X and Y are not near-standard.

Definition. By a Π_1^0 set in ${}^*M$ we mean a set of the form $\bigcap_n A_n$ where A_n is a chain of internal subsets of ${}^*M$.

The following fact is a consequence of ω_1-saturation.

1.6 (Henson [12]). *If A_n is a chain of internal subsets of ${}^*M$, then* $\mathrm{st}(\bigcap_n A_n) = \bigcap_n \mathrm{st}(A_n)$. *If A is a Π_1^0 set in ${}^*M$, then* $\mathrm{st}(A)$ *is closed.*

Here is an analogous result for compact sets.

Proposition 1.7. *If A is a Π_1^0 set in ${}^*M$ and $A \subseteq \mathrm{ns}({}^*M)$, then* $\mathrm{st}(A)$ *is compact.*

Proof. Suppose $B = \mathrm{st}(\bigcap_n A_n)$ where A_n is a chain of internal sets and $\bigcap_n A_n \subseteq \mathrm{ns}({}^*M)$. To show B is compact, it suffices to show that every sequence $\langle b_m : m \in \mathbb{N}\rangle$ of elements of B has a subsequence converging to an element of B. For each $m \in \mathbb{N}$, choose $a_m \in \bigcap_n A_n$ such that $\mathrm{st}(a_m) = b_m$, and use ω_1-saturation to extend this to an internal sequence $\langle a_m : m \in {}^*N \rangle$. Let U_k be a countable open basis for M. Whenever $b_m \in U_k$ we have $a_m \in {}^*U_k$. By ω_1-saturation there is an infinite hyperinteger H such that $a_H \in \bigcap_n A_n$, and $a_H \in {}^*U_k$ whenever all but finitely many b_m belong to U_k. Then a_H is near-standard and $b = \mathrm{st}(a_H)$ belongs to B. Moreover, whenever $b \in U_k$, $a_H \in {}^*U_k$ and thus for any U_j disjoint from U_k, there are infinitely many $m \in \mathbb{N}$ with $b_m \notin U_j$. Therefore b is a limit of a subsequence of $\langle b_m \rangle$, as required. □

Definition. Let $f: M \to N$ where N and M are Polish spaces. A *lifting* of f is an internal function $F: {}^*M \to {}^*N$ such that whenever $x \in M$, $X \in {}^*M$, and $\mathrm{st}(X) = x$, we have $\mathrm{st}(F(X)) = f(x)$.

1.8 (Robinson [27]). *If $f: M \to N$ has a lifting, then f is continuous. If f is continuous, then ${}^*f$ is a lifting of f.*

Definition. A set A of hyperintegers is said to *contain all sufficiently small infinite J* iff there is an infinite hyperinteger H such that $J \in A$ for all infinite hyperintegers $J \leq H$.

The *overspill* principle (see [27]) states that an internal set of hyperintegers contains all sufficiently large $n \in \mathbb{N}$ if and only if it contains all sufficiently small infinite J.

We conclude this section with a well known general consequence of ω_1-saturation which we shall often find useful.

1.9. *If for each $n \in \mathbb{N}$, A_n is a set of hyperintegers which contains all sufficiently small infinite J, then $\bigcap_n A_n$ contains all sufficiently small infinite J.*

Proof. For each $n \in \mathbb{N}$, there is an infinite hyperinteger H_n such that $J \in A_n$ for all infinite $J \leq H_n$. By ω_1-saturation, there is an infinite hyperinteger H such that $H \leq H_n$ for all $n \in \mathbb{N}$. Then $J \in \bigcap_n A_n$ for all infinite $J \leq H$. □

2. Preliminaries on hyperfinite probability spaces

Throughout this paper, (Ω, P) will be a *hyperfinite probability space*. That is, a probability space (Ω, P) where Ω is a hyperfinite set, $\bar{P}$ is the internal counting probability measure on Ω, and P is the associated Loeb measure. We sometimes abuse notation and denote the space (Ω, P) by Ω. Borel(Ω) is the σ-algebra generated by the set of internal subsets of Ω.

The *internal counting probability measure* $\bar{P}$ on Ω is defined by $\bar{P}(A) = \#(A)/\#(\Omega)$ for each internal $A \subseteq \Omega$, where $\#(A)$ is the internal cardinality of A. The *Loeb measure* P on Ω is defined as the completion of the unique countably additive probability measure on Borel(Ω) such that $P(A) = \mathrm{st}(\bar{P}(A))$ for each internal set $A \subseteq \Omega$. Its existence is proved by Loeb [23].

If $X : \Omega \to {}^*\mathbb{R}$ is internal, $\bar{E}[X]$ denotes the expected value of X with respect to $\bar{P}$, that is,

$$\bar{E}[X] = \sum_{\omega \in \Omega} X(\omega)\, \bar{P}(\{\omega\}).$$

If $x : \Omega \to \mathbb{R}$ is P-integrable, $E[x]$ denotes the expected value, or integral, of x with respect to P.

Definition. Let RV(Ω, $^*M$) be the internal set of all internal functions $X : \Omega \to {}^*M$ with the internal *metric

$$\bar{\rho}(X, Y) = {}^*\inf\{\varepsilon \in {}^*\mathbb{R} : \bar{P}[{}^*\rho(X, Y) \leq \varepsilon] \geq 1 - \varepsilon\}.$$

We write $X \approx Y$ if $\bar{\rho}(X, Y) \approx 0$.

$\bar{\rho}$ is a *metric if every nonempty internal subset of Ω has positive $\bar{P}$-measure. Otherwise it is only a *pseudometric.

If C is a subset of M and $a \in M$, $\rho(a, C)$ is defined as the infimum of $\rho(a, b)$ for $b \in C$. If $\varepsilon > 0$, C^{ε} denotes the set of all $a \in M$ such that $\rho(a, C) \leq \varepsilon$. If A is an internal subset of RV(Ω, $^*M$) and $X \in \mathrm{RV}(\Omega, {}^*M)$, then $\bar{\rho}(X, A)$ is defined as the *infimum of $\bar{\rho}(X, Y)$ for $Y \in A$, and A^{ε} denotes the internal set of all x such that $\bar{\rho}(X, A) \leq \varepsilon$.

We shall now begin to develop an analogy between the pair ($^*M$, M) and the pair (RV(Ω, $^*M$), rv(Ω, M)). If M has more than one point, the metric space rv(Ω, M) is not separable (see Example 3.9). However, we shall see that ω_1-saturation will often compensate for the lact of separability. We first consider the analogue of the standard part function.

Definition. An internal function $X \in \mathrm{RV}(\Omega, {}^*M)$ will be called *near-standard*, $X \in \mathrm{ns}(\Omega, {}^*M)$, iff $X(\omega)$ is near-standard for P-almost all ω. We define the *standard part function* $\mathrm{st} : \mathrm{ns}(\Omega, {}^*M) \to \mathrm{rv}(\Omega, M)$ by letting st(X) be the equivalence class of a random variable $x : \Omega \to M$ such that $x(\omega) = \mathrm{st}(X(\omega))$ for P-almost all ω. If $A \subseteq \mathrm{RV}(\Omega, {}^*M)$, we define $\mathrm{st}(A) = \{\mathrm{st}(X) : X \in A\}$. If $B \subseteq \mathrm{rv}(\Omega, M)$, we define $\mathrm{st}^{-1}(B) = \{X \in \mathrm{ns}(\Omega, {}^*M) : \mathrm{st}(X) \in B\}$.

Following Anderson [3], we say that X is a *lifting* of x if $\mathrm{st}(X) = x$. Notice that we are now using lifting in two different senses: the topological notion of a lifting of a function $f : M \to N$ between Polish spaces, and the measure-theoretic notion of a lifting of a function $x : \Omega \to M$. Sometimes the former notion is called a uniform lifting because it works everywhere instead of almost everywhere. In this paper we shall simply use the word lifting in both cases and depend on the context to distinguish the two notions.

2.1 (Anderson [3]). *A function $x:\Omega\to M$ has a lifting if and only if x is P-measurable.*

Remarks. (a) If M is compact, then by 1.5(d), $\mathrm{ns}(\Omega, {}^*M) = \mathrm{RV}(\Omega, {}^*M)$.

(b) For all $X, Y \in \mathrm{RV}(\Omega, {}^*M)$, $X \approx Y$ if and only if $X(\omega) \approx Y(\omega)$ for P-almost all ω.

(c) If $X \in \mathrm{ns}(\Omega, {}^*M)$, then $X \approx Y$ if and only if $Y \in \mathrm{ns}(\Omega, {}^*M)$ and $\mathrm{st}(X) = \mathrm{st}(Y)$.

(d) By (b) and 1.5(e), the set $\mathrm{ns}(\Omega, {}^*M)$ and function $\mathrm{st}:\mathrm{ns}(\Omega, {}^*M)\to \mathrm{rv}(\Omega, M)$ depend only on the topology of M and not on the metric ρ.

(e) We shall often use the following consequence of ω_1-saturation. Each sequence $\langle X_n : n \in \mathbb{N}\rangle$ of internal elements of $\mathrm{RV}(\Omega, {}^*M)$ can be extended to an internal sequence $\langle X_J : J \in {}^*\mathbb{N}\rangle$ of elements of $\mathrm{RV}(\Omega, {}^*M)$.

(f) For each sequence $\langle x_n : n \in \mathbb{N}\rangle$ of elements of $\mathrm{rv}(\Omega, M)$ there exists an internal sequence $\langle X_J : J \in {}^*\mathbb{N}\rangle$ of elements of $\mathrm{RV}(\Omega, {}^*M)$ such that X_n lifts x_n for all $n \in \mathbb{N}$. We shall call $\langle X_J : J \in {}^*\mathbb{N}\rangle$ a *lifting* of $\langle x_n : n \in \mathbb{N}\rangle$.

Proposition 2.2. *Suppose $X, Y \in \mathrm{ns}(\Omega, {}^*M)$, $x = \mathrm{st}(\mathrm{X})$, and $y = \mathrm{st}(Y)$. Then $\hat\rho(x, y) = \mathrm{st}(\bar\rho(X, Y))$.*

Proof. Since the metric ρ is continuous on $M \times M$, ${}^*\rho(X(\omega), Y(\omega))$ as a function from Ω into ${}^*\mathbb{R}$ is a lifting of $\rho(x(\omega), y(\omega))$. We see from the definitions that for all positive real $\varepsilon > \delta$, we have

$$\hat\rho(x, y) \leqslant \varepsilon \quad \text{implies} \quad \bar\rho(X, Y) \leqslant \delta,$$

$$\bar\rho(X, Y) \leqslant \varepsilon \quad \text{implies} \quad \hat\rho(x, y) \leqslant \delta.$$

Therefore $\hat\rho(x, y) \approx \bar\rho(X, Y)$. □

Here is a characterization of a limit for the space $\mathrm{rv}(\Omega, M)$. This characterization depends only on the topology of M and not on the metric ρ.

Proposition 2.3. *Suppose that $X \in \mathrm{RV}(\Omega, {}^*M)$ lifts $x \in \mathrm{rv}(\Omega, M)$, and $\langle X_n : n \in {}^*\mathbb{N}\rangle$ lifts $\langle x_N : n \in \mathbb{N}\rangle$. Then $\lim_{n\to\infty} x_n = x$ if and only if $X_J \approx X$ holds for all sufficiently small infinite J.*

Proof. First suppose $\lim_{n\to\infty} x_n = x$. By overspill, for each $m \in \mathbb{N}$, $\bar\rho(X_J, X) \leqslant 1/m$ for all sufficiently small infinite J. Then by 1.9, $X_J \approx X$ for all sufficiently small infinite J.

Now assume that $X_J \approx X$ for all sufficiently small infinite J. Then for each real $\varepsilon > 0$, $\bar\rho(X_J X) \leqslant \varepsilon$ for all sufficiently small infinite J. By overspill, $\bar\rho(X_n, X) \leqslant \varepsilon$, and hence $\hat\rho(x_n, x) \leqslant \varepsilon$, for all sufficiently large $n \in \mathbb{N}$. This proves that $\lim_{n\to\infty} x_n = x$. □

Define the internal function $\mathrm{LAW}: \mathrm{RV}(\Omega, {}^*M) \to {}^*\mathrm{Meas}(M)$ by

$$\mathrm{LAW}(X)(C) = \bar{P}(X^{-1}(C))$$

for each *Borel subset C of ${}^*M$. Notice that for each X, the *measure $\mathrm{LAW}(X)$ is concentrated on the range of X, which is a hyperfinite subset of ${}^*M$ of size at most the size of Ω.

Proposition 2.4 (Loeb [24]). *Suppose* $X \in \mathrm{ns}(\Omega, {}^*M)$ *and* $x = \mathrm{st}(X)$. *Then in the space* $\mathrm{Meas}(M)$, $\mathrm{law}(x) = \mathrm{st}(\mathrm{LAW}(X))$.

Proof. The family of finite intersections of sets of the form $\{\mu: \int \psi \, \mathrm{d}\mu \in (a, b)\}$, where ψ is a bounded continuous function $\psi: M \to \mathbb{R}$, forms an open basis for $\mathrm{Meas}(M)$. For any such ψ, we have

$$\int \psi \, \mathrm{d}(\mathrm{law}(x)) = E[\psi(x)] \approx \bar{E}[({}^*\psi)(X)] = \int ({}^*\psi) \, \mathrm{d}(\mathrm{LAW}(X)).$$

Thus, if $\int \psi \, \mathrm{d}(\mathrm{law}(x)) \in (a, b)$, then $\int ({}^*\psi) \, \mathrm{d}(\mathrm{LAW}(X)) \in {}^*(a, b)$. It follows that $\mathrm{LAW}(X)$ belongs to the star of any open neighborhood of $\mathrm{law}(x)$, that is, $\mathrm{Law}(X) \approx \mathrm{law}(x)$. □

Proposition 2.5. (a) (Universality) *The function* $\mathrm{law}: \mathrm{rv}(\Omega, M) \to \mathrm{Meas}(M)$ *is onto.*

(b) (Homogeneity) *If* $x, y \in \mathrm{rv}(\Omega, M)$, *then* $\mathrm{law}(x) = \mathrm{law}(y)$ *if and only if there is an internal permutation* h *of* Ω *such that* $x(h\omega) = y(\omega)$ *P-almost surely.*

(c) (Saturation) *For any* $x \in \mathrm{rv}(\Omega, M)$ *and* $\bar{x}, \bar{y} \in \mathrm{rv}(\Gamma, M)$ *such that* $\mathrm{law}(x) = \mathrm{law}(\bar{x})$, *there exists* $y \in \mathrm{rv}(\Omega, M)$ *such that* $\mathrm{law}(x, y) = \mathrm{law}(\bar{x}, \bar{y})$.

A proof of 2.5 can be found in [20].

Proposition 2.6. (a) $\mathrm{ns}({}^*M)$ *is a countable intersection of countable unions of internal sets.*

(b) $\mathrm{ns}(\Omega, {}^*M)$ *is a countable intersection of countable unions of internal sets.*

Proof. (a) Let $\{a_n: n \in \mathbb{N}\}$ be a countable dense subset of M, and let A_{kn} be the set of all points within distance $1/k$ of $\{a_1, \ldots, a_n\}$. Then

$$\mathrm{ns}({}^*M) = \bigcap_k \bigcup_n {}^*A_{kn}.$$

(b) Let A_{kn} be defined as in part (a), and let B_{kn} be the set of all internal $X: \Omega \to {}^*M$ such that $\bar{P}[X \in {}^*A_{kn}] \geq 1 - 1/k$. Then each B_{kn} is internal, and

$$\mathrm{ns}(\Omega, {}^*M) = \bigcap_k \bigcup_n B_{kn}. \qquad \square$$

Proposition 2.7. $X \in \mathrm{ns}(\Omega, {}^*M)$ *if and only if* $\mathrm{LAW}(X) \in \mathrm{ns}({}^*\mathrm{Meas}(M))$.

Proof. The implication from left to right follows from 2.4. Suppose LAW(X) is near-standard. Let $\mu = \mathrm{st}(\mathrm{LAW}(X))$. Then for any bounded continuous function $\psi : M \to \mathbb{R}$, we have

$$\int (^*\psi)\,\mathrm{d}\,\mathrm{LAW}(X) \approx \int \psi\,\mathrm{d}\mu.$$

It follows that for each compact set $K \subseteq M$,

$$\bar{P}[X \in {}^*(K^{1/n})] \geqslant \mu(K) - 1/n \quad \text{for all} \quad n \in \mathbb{N},$$

and since ${}^*K \subseteq \mathrm{st}^{-1}(K)$,

$$P[X \in \mathrm{st}^{-1}(\mathrm{K})] \geqslant \mu(K).$$

By Prohorov's Theorem, each singleton $\{\mu\}$ is tight, so for each $n \in \mathbb{N}$ there is a compact set K_n such that $\mu(K_n) \geqslant 1 - 1/n$. Therefore $X(\omega)$ is near-standard for P-almost all ω, and $X \in \mathrm{ns}(\Omega, {}^*M)$. □

Proposition 2.8. *Let A_n be a decreacing chain of internal subsets of* $\mathrm{RV}(\Omega, {}^*M)$.

(a) $\mathrm{st}(\bigcap_n A_n) = \bigcap_n \mathrm{st}(A_n)$.

(b) $\mathrm{st}(\bigcap_n A)$ *is closed.*

(c) $\mathrm{st}^{-1}(\mathrm{st}(\bigcap_n A_n)) = \bigcap_n (A_n)^{1/n} \cap \mathrm{ns}(\Omega, {}^*M)$.

(d) *If* $\bigcap_n A_n \subset \mathrm{ns}(\Omega, {}^*M)$, *then* $\bigcap_n (A_n)^{1/n} \subseteq \mathrm{ns}(\Omega, {}^*M)$.

Proof. (a) It is trivial that $\mathrm{st}(\bigcap_n A_n) \subseteq \bigcap_n \mathrm{st}(A_n)$. Suppose $x \in \bigcap_n \mathrm{st}(A_n)$. Let X lift x. For each n there exists $Y \in A_n$ such that $\bar{\rho}(X, Y) \leqslant 1/n$. By ω_1-saturation there exists $Y \in \bigcap_n A_n$ such that $X \approx Y$, so $x = \mathrm{st}(Y) \in \mathrm{st}(\bigcap_n A_n)$.

(b) By (a), it suffices to prove that $\mathrm{st}(A)$ is closed for each internal $A \subseteq \mathrm{RV}(\Omega, {}^*M)$. Let $x \notin \mathrm{st}(A)$ and let X lift x. Then $\bar{\rho}(X, Y)$ is noninfinitesimal for all $Y \in A$. By overspill, there is an $m \in \mathbb{N}$ such that $\bar{\rho}(X, Y) \geqslant 1/m$ for all $Y \in A$. Then by Proposition 2.2, $\hat{\rho}(x, y) \geqslant 1/m$ for all $y \in \mathrm{st}(A)$. Therefore the complement of $\mathrm{st}(A)$ is open, and $\mathrm{st}(A)$ is closed.

(c) The left side is trivially included in the right side. Let X belong to the right side. Then X has a standard part x. By ω_1-saturation there exists Y such that for all n, $Y \in A_n$ and $\bar{\rho}(X, Y) \leqslant 1/n$. Then $X \approx Y$, so $x = \mathrm{st}(Y) \in \mathrm{st}(\bigcap_n A_n)$.

(d) Let $X \in \bigcap_n (A_n)^{1/n}$. For each $n \in \mathbb{N}$ there exists $Y \in A_n$ such that $\bar{\rho}(X, Y) \leqslant 1/n$. Since the A_n are decreasing, it follows by ω_1-saturation that there exists Y such that for all $n \in \mathbb{N}$, $Y \in A_n$ and $\bar{\rho}(X, Y) \leqslant 1/n$. Then $X \approx Y$ and $Y \in \bigcap_n A_n$. Therefore $Y \in \mathrm{ns}(\Omega, {}^*M)$ and hence $X \in \mathrm{ns}(\Omega, {}^*M)$. □

3. Tight sets

Here is a characterization of tight subsets of $\mathrm{rv}(\Omega, M)$ where (Ω, P) is a hyperfinite probability space.

Theorem 3.1. *Let (Ω, P) be a hyperfinite probability space, and let $T \subseteq \mathrm{rv}(\Omega, M)$. The following are equivalent:*

(a) *T is tight.*

(b) *$T \subseteq \mathrm{st}(A)$ for some internal set $A \subseteq \mathrm{ns}(\Omega, {}^*M)$.*

(c) *$T \subseteq \mathrm{st}(A)$ for some Π_1^0 set $A \subseteq \mathrm{ns}(\Omega, {}^*M)$.*

Proof. First assume (a). We shall prove (b). By Prohorov's Theorem, there is a compact set $K \subseteq \mathrm{Meas}(M)$ such that $\{\mathrm{law}(x) : x \in T\} \subseteq K$. Let π be a metric for $\mathrm{Meas}(M)$. By 2.5(a) (Universality), for each $\mu \in K$ there exists $y \in \mathrm{rv}(\Omega, M)$ with $\mathrm{law}(y) = \mu$. If Y lifts y, then $\mathrm{law}(y) = \mathrm{st}(\mathrm{LAW}(Y))$ by 2.4, and thus ${}^*\pi(\mathrm{LAW}(Y), {}^*\mu) \approx 0$. Since K is compact, each element of ${}^*K$ is near-standard. Thus each $n \in \mathbb{N}$ has the property that for each $\nu \in {}^*K$ there exists $Y \in \mathrm{RV}(\Omega, {}^*M)$ with ${}^*\pi(\mathrm{LAW}(Y), \nu) \leqslant 1/n$. By overspill there is an infinite H such that for each $\nu \in {}^*K$ there exists $Y \in \mathrm{RV}(\Omega, {}^*M)$ with ${}^*\pi(\mathrm{LAW}(Y), \nu) \leqslant 1/H$. Let A be the internal set of all $Y \in \mathrm{RV}(\Omega, {}^*M)$ such that for some $\nu \in {}^*K$, ${}^*\pi(\mathrm{LAW}(Y), \nu) \leqslant 1/H$. For each $Y \in A$, $\mathrm{LAW}(Y)$ is near-standard, and by 2.7, Y is near-standard. Thus $A \subseteq \mathrm{ns}(\Omega, {}^*M)$. Let $x \in T$. Then $\mathrm{law}(x) \in K$, and hence there exists $Z \in A$ with ${}^*\pi(\mathrm{LAW}(Z), {}^*\mathrm{law}(x)) \leqslant 1/H$. Let $z = \mathrm{st}(Z)$. By 2.4, $\mathrm{law}(z) = \mathrm{st}(\mathrm{LAW}(Z))$, and so $\mathrm{law}(z) = \mathrm{law}(x)$. By 2.5(b) (homogeneity), we have $x(\omega) = z(h\omega)$ P-almost surely for some internal permutation h of Ω. Let $X \in \mathrm{RV}(\Omega, {}^*M)$ be given by $X(\omega) = Z(h\omega)$. Since h is internal, it preserves $\bar{P}$ and P. Therefore X is a lifting of x. Moreover, $\mathrm{LAW}(X) = \mathrm{LAW}(Z)$. We conclude that $X \in A$ and $T \subseteq \mathrm{st}(A)$, and hence (b) holds.

The implication from (b) to (c) is trivial.

Now assume (c), that $T \subseteq \mathrm{st}(A)$ for some Π_1^0 set $A = \bigcap_n A_n \subseteq \mathrm{ns}(\Omega, {}^*M)$. From the proof of Proposition 2.6, there is a chain of compact (in fact finite) sets $K_m \subseteq M$ such that

$$\mathrm{ns}(\Omega, {}^*M) = \bigcap_j \bigcup_m B_{jm}$$

where B_{jm} is the internal set

$$B_{jm} = \{X \in \mathrm{RV}(\Omega, {}^*M) : \bar{P}[X \in {}^*(K_m^{1/j})] \geqslant 1 - 1/j\}.$$

For each $j \in \mathbb{N}$ we have

$$\bigcap_n A_n \subseteq \bigcup_m B_{jm}.$$

By ω_1-saturation, for each j there exist $m(j)$ and $n(j) \in \mathbb{N}$ such that $A_{n(j)} \subseteq B_{j,m(j)}$. Then for each j, $A \subseteq B_{j,m(j)}$. Now fix $k \in \mathbb{N}$. Let $h(j) = m(k \cdot 2^j)$. For each $X \in A$, we have

$$\bar{P}[X \in {}^*(K_{h(j)}^{1/j})] \geqslant 1 - k^{-1} \cdot 2^{-j}.$$

Therefore for each $x \in T$,

$$P[x \in (K_{h(j)}^{1/j})] \geqslant 1 - k^{-1} \cdot 2^{-j},$$

and

$$P[x \in K] \geqslant 1 - 1/k$$

where $K = \bigcap_j (K_{h(j)}^{1/j})$. K is closed and totally bounded, and therefore compact. This shows that T is tight. □

A sequence $\langle x_n : n \in \mathbb{N} \rangle$ in $\mathrm{rv}(\Omega, M)$ is said to be *tight* iff its range is a tight set. Here is a characterization of tight sequences in terms of liftings.

Theorem 3.2. *Let $\langle x_n : n \in \mathbb{N} \rangle$ be a sequence in* $\mathrm{rv}(\Omega, M)$ *and let $\langle X_J : J \in {}^*\mathbb{N} \rangle$ be a lifting of $\langle x_n : n \in \mathbb{N} \rangle$. Then $\langle x_n : n \in \mathbb{N} \rangle$ is tight if and only if X_J is near-standard for all sufficiently small infinite J.*

Proof. Suppose $\langle x_n \rangle$ is tight. By Theorem 3.1, there is an internal set $A \subseteq \mathrm{ns}(\Omega, {}^*M)$ such that for each $n \in \mathbb{N}$, x_n has a lifting $Y_n \in A$. Since X_n is also a lifting of x_n, we have $\bar{\rho}(X_n, Y_n) \approx 0$, and hence $\bar{\rho}(X_n, Y_n) \leqslant 1/n$ for all $n \in \mathbb{N}$. By overspill, there is an infinite H such that $Y_J \in A$ and $\bar{\rho}(X_J, Y_J) \leqslant 1/J$ for all $J \leqslant H$. Then X_J is near-standard for all $J \leqslant H$.

Now suppose X_J is near-standard for all infinite $J \leqslant H$. Then each x_n belongs to $\mathrm{st}(A)$ where A is the internal set $\{X_J : J \leqslant H\}$, and $A \subseteq \mathrm{ns}(\Omega, {}^*M)$. □

We now introduce the notions of neoclosed and neocompact sets, which play roles for the space $\mathrm{rv}(\Omega, M)$ analogous to the closed and compact subsets of M.

Definition. By a Π_1^0 set (in $\mathrm{RV}(\Omega, {}^*M)$) we mean a set of the form $\bigcap_n A_n$ where A_n is a decreasing chain of internal subsets of $\mathrm{RV}(\Omega, {}^*M)$. We say that a subset C of $\mathrm{rv}(\Omega, M)$ is *neoclosed* iff $C = \mathrm{st}(A)$ for some Π_1^0 set $A \subseteq \mathrm{RV}(\Omega, {}^*M)$. C is said to be *neocompact* iff C is tight and neoclosed.

Corollary 3.3. (a) *Every neoclosed set is closed.*

(b) *C is neoclosed if and only if there is a Π_1^0 set B such that* $\mathrm{st}^{-1}(C) = B \cap \mathrm{ns}(\Omega, {}^*M)$.

Proof. By Proposition 2.8. □

Proposition 3.4. *The set of neoclosed subsets of* $\mathrm{rv}(\Omega, M)$ *is closed under finite unions and under countable intersections.*

Proof. It is easily seen that the family of Π_1^0 sets is closed under finite unions and countable intersections. The union of two neoclosed sets is neoclosed because $\mathrm{st}(A \cup B) = \mathrm{st}(A) \cup \mathrm{st}(B)$. Let C_n be neoclosed for each $n \in \mathbb{N}$. By 3.3(b), for each n there is a Π_1^0 set B_n such that

$$\mathrm{st}^{-1}(C_n) = B_n \cap \mathrm{ns}(\Omega, {}^*M).$$

Then

$$\mathrm{st}^{-1}\Big(\bigcap_n C_n\Big) = \bigcap_n B_n \cap \mathrm{ns}(\Omega, {}^*M),$$

and

$$\bigcap_n C_n = \mathrm{st}\Big(\bigcap_n B_n\Big).$$

Since countable intersections of Π_1^0 sets are Π_1^0, $\bigcap_n B_n$ is Π_1^0 and thus $\bigcap_n C_n$ is neoclosed. □

We now turn to the neocompact sets. If M is a compact metric space, then the set $\mathrm{rv}(\Omega, M)$ is neocompact. Here is a characterization of neocompact sets.

Corollary 3.5. *Let $C \subseteq \mathrm{rv}(\Omega, M)$, The following are equivalent.*

(a) *C is neocompact.*

(b) *$\mathrm{st}^{-1}(C)$ is a Π_1^0 set.*

(c) *$C = \mathrm{st}(A)$ for some Π_1^0 set $A \subseteq \mathrm{ns}(\Omega, {}^*M)$.*

Proof. (a) implies (b): Let C be neocompact. By 3.3 and 3.1, there are Π_1^0 sets A and B such that $A \subseteq \mathrm{ns}(\Omega, {}^*M)$, $\mathrm{st}^{-1}(C) \subseteq A$, and $\mathrm{st}^{-1}(C) = B \cap \mathrm{ns}(\Omega, {}^*M)$. Then $\mathrm{st}^{-1}(C) = A \cap B$, and $A \cap B$ is a Π_1^0 set.

The implication from (b) to (c) holds with $A = \mathrm{st}^{-1}(C)$.

(c) implies (a): If (c) holds, then C is neoclosed by definition and C is tight by 3.1. □

Corollary 3.6. *Every tight set $T \subseteq \mathrm{rv}(\Omega, M)$ is contained in a neocompact set.*

Proof. By 3.1 and 3.5. □

The following result is a countable compactness property for neocompact sets.

Proposition 3.7. *The intersection of a countable chain of nonempty neocompact subsets of $\mathrm{rv}(\Omega, M)$ is nonempty.*

Proof. Let C_n, $n \in \mathbb{N}$, be a countable chain of nonempty neocompact sets. For each n, $\mathrm{st}^{-1}(C_n) = \bigcap_k A_{kn}$ where each A_{kn} is internal. The family $\{A_{kn}: k, n \in \mathbb{N}\}$ has the finite intersection property. By ω_1-saturation,

$$\mathrm{st}^{-1}\Big(\bigcap_n C_n\Big) = \bigcap_n \bigcap_k A_{kn}$$

is nonempty, so $\bigcap_n C_n$ is nonempty. □

Corollary 3.8. *Let C_n be a countable decreasing chain of neoclosed subsets of $\mathrm{rv}(\Omega, M)$ and let $\langle x_n \rangle$ be a tight sequence in $\mathrm{rv}(\Omega, M)$. If $x_n \in C_n$ for each $n \in \mathbb{N}$, then $\bigcap_n C_n$ is nonempty.*

Proof. By 3.6, there is a noecompact set B containing each x_n. By 3.4, $B \cap C_n$ is neocompact for each n. $B \cap C_n$ is nonempty because it contains the point x_n. By 3.7, $\bigcap_n (B \cap C_n)$ is nonempty. $\square$

Given two or more spaces of random variables, say $\mathrm{rv}(\Omega, M)$ and $\mathrm{rv}(\Gamma, N)$ where Ω and Γ are hyperfinite probability spaces, the notions of a tight relation and a neoclosed relation on $\mathrm{rv}(\Omega, M) \times \mathrm{rv}(\Gamma, N)$ are defined in the natural way. All of the results in this section readily extend to relations.

We conclude this section with an example illustrating the difference between compact and neocompact.

Example 3.9. Let Ω be a hyperfinite probability space. The elements of $\mathrm{rv}(\Omega, \{0, 1\})$ are the characteristic functions of Loeb measurable sets, which we shall identify with the sets themselves. The distance between two elements $x, y \in \mathrm{rv}(\Omega, \{0, 1\})$ is the Loeb measure $P(x \bigtriangleup y)$ of their symmetricx difference. For each internal $X \subseteq \Omega$, X is near-standard and $\mathrm{st}(X)$ is the equivalence class of X, i.e., the set of all $x \subseteq \Omega$ such that $P(X \bigtriangleup x) = 0$. The set $\mathrm{rv}(\Omega, \{0, 1\})$ is obviously tight and neoclosed and thus neocompact. Now let $\{x_i : i \in I\}$ be a maximal subset of $\mathrm{rv}(\Omega, \{0, 1\})$ such that $P(x_i \bigtriangleup x_j) \geqslant 1/2$ whenever $i \neq j$. I is infinite because for any finite subset $S \subseteq \mathrm{rv}(\Omega, \{0, 1\})$, there is an $x \in \mathrm{rv}(\Omega, \{0, 1\})$ which is at distance $1/2$ from each $y \in S$. For each $i \in I$, the set $B_i = \{y : P(x_i \bigtriangleup y) \geqslant 1/2\}$ is neoclosed, because if X_i lifts x_i, then B_i is the standard part of the Π_1^0 set $\bigcap_n \{Y : \bar{P}(X_i \bigtriangleup Y) \geqslant 1/2 - 1/n\}$. Let J_n, $n \in \mathbb{N}$, be an increasing chain of proper subsets of I such that $\bigcup_n J_n = I$. Then for each n, the set $C_n = \bigcap_{i \in J_n} B_i$ is closed and nonempty. The maximality of the family $\{x_i : i \in I\}$ implies that $\bigcap_n C_n = 0$. We may now draw the following conclusions:

(a) C_n is a countable chain of nonempty tight closed sets with empty intersection.

(b) The set $\mathrm{rv}(\Omega, \{0, 1\})$ is neocompact but not compact.

(c) For some n, the set C_n is tight and closed but not neocompact.

(d) The metric space $\mathrm{rv}(\Omega, \{0, 1\})$ is not separable (e.g., by (b) and Prohorov's Theorem).

4. Liftable functions

We now introduce the notion of a liftable function which playes a role in our setting analogous to the continuous functions. The liftable functions will be a class of functions from $\mathrm{rv}(\Omega, M)$ into $\mathrm{rv}(\Gamma, N)$, where Γ is another hyperfinite probability space and N is another Polish space. We also wish to include the case of functions from $\mathrm{rv}(\Omega, M)$ into N. For this reason we allow Γ to be either finite or hyperfinite, and identify N with the space $\mathrm{rv}(1, N)$ where 1 is a one-element probability space. By 1.5(b), a subset of a Polish space N is neoclosed if and only if it is closed. To define the liftable functions, we shall introduce an analogue of the notion of a lifting which applies to functions from $\mathrm{rv}(\Omega, M)$ into $\mathrm{rv}(\Gamma, N)$, and then call a function liftable iff it has a lifting.

Throughout this section, L, M and N are Polish spaces, Λ, Ω and Γ are finite or hyperfinite probability spaces with counting measures, and ρ is a metric for M.

Definition. A *lifting* of a function $f: \mathrm{rv}(\Omega, M) \to \mathrm{rv}(\Gamma, N)$ is an internal function $F: \mathrm{RV}(\Omega, {}^*M) \to \mathrm{RV}(\Gamma, {}^*N)$ such that whenever $X \in \mathrm{RV}(\Omega, {}^*M)$ lifts $x \in \mathrm{rv}(\Omega, M)$, $F(X)$ lifts $f(x)$. A function f is *liftable* iff f has a lifting.

Examples 4.1. (a) The identity function on $\mathrm{RV}(\Omega, {}^*M)$ is a lifting of the identity function on $\mathrm{rv}(\Omega, M)$. Hence the identity function on $\mathrm{rv}(\Omega, M)$ is liftable.

(b) By 1.8, a function $f: M \to N$ from a Polish space M into another Polish space N is liftable if and only if it is continuous.

(c) Proposition 2.2 shows that the function

$$\bar{\rho}: \mathrm{RV}(\Omega, {}^*M \times {}^*M) \to {}^*\mathbb{R}$$

is a lifting of the distance function

$$\hat{\rho}: \mathrm{rv}(\Omega, M \times M) \to \mathbb{R}.$$

Thus $\hat{\rho}$ is liftable.

(d) Proposition 2.4 shows that the function

$$\mathrm{LAW}: \mathrm{RV}(\Omega, {}^*M) \to {}^*\mathrm{Meas}(M)$$

is a lifting of the law function

$$\mathrm{law}: \mathrm{rv}(\Omega, M) \to \mathrm{Meas}(M).$$

Thus the function law is liftable.

We now study the connection between liftable functions, neoclosed sets, and tight sets.

Proposition 4.2. *A function $f: \mathrm{rv}(\Omega, M) \to N$ is liftable if and only if for every closed subset C of N, $f^{-1}(C)$ is neoclosed.*

Proof. Let F be a lifting of f. Let C be a closed subset of N. Let π be a metric for N. Assume that X lifts x. Then $F(X)$ lifts $f(x)$. Thus

$$f(x) \in C \quad \text{iff} \quad F(x) \in \mathrm{st}^{-1}(C) \quad \text{iff} \quad X \in \bigcap_n A_n$$

where $A_n = \{Y: F(Y) \in {}^*(C^{1/n})\}$. Therefore $f^{-1}(C) = \mathrm{st}(\bigcap_n A_n)$ is neoclosed.

Now suppose $f^{-1}(C)$ is neoclosed for every closed $C \subseteq N$. Take a countable closed base C_n for N. For each n, let A_{kn} be a decreasing chain of internal subsets of $\mathrm{RV}(\Omega, {}^*M)$ such that $f^{-1}(C_n) = \mathrm{st}(\bigcap_k A_{kn})$. By Proposition 2.8,

$$\mathrm{st}\Big(\bigcap_k A_{kn}\Big) = \mathrm{st}\Big(\bigcap_k (A_{kn})^{1/k}\Big),$$

$$\mathrm{st}^{-1}\Big(\mathrm{st}\Big(\bigcap_k A_{kn}\Big)\Big) = \bigcap_k (A_{kn})^{1/k} \cap \mathrm{ns}(\Omega, {}^*M).$$

We claim that there is an internal function F such that for all k and n,

$$F^{-1}({}^*C_n) \subseteq (A_{kn})^{1/k}.$$

To see this, let $k \in \mathbb{N}$. For each subset U of $\{1, \ldots, k\}$, choose $a_U \in M - \bigcup_{n\in U} C_n$ if $M - \bigcup_{n\in U} C_n \neq \emptyset$, and choose $a_U \in M$ arbitrarily otherwise. Let F_k be the function defined by $F_k(X) = a_U$ where $U = \{n \leq k : x \notin (A_{nk})^{1/k}\}$. Then F_k is internal and for all $m, n \leq k$,

$$F_k^{-1}({}^*C_n) \subseteq (A_{kn})^{1/k}.$$

It follows by ω_1-saturation that the required function F exists.

We now show that F is a lifting of f. Suppose X lifts x and $f(x)$ belongs to a basic open set B. Let C_n be the complement of B. Then $x \notin f^{-1}(C_n)$, and hence there is a k such that $x \notin \mathrm{st}((A_{kn})^{1/k})$. Then $X \notin (A_{kn})^{1/k}$, and so $F(X) \notin {}^*C_n$ and $F(X) \in {}^*B$. This shows that $F(X)$ lifts $f(x)$, so F lifts f. □

Since all neoclosed sets are closed, it follows from 4.2 that all liftable functions $f : \mathrm{rv}(\Omega, M) \to N$ are continuous. We now prove one direction of the analogue of Proposition 4.2 for functions whose values are random variables.

Proposition 4.3. *Suppose $f : \mathrm{rv}(\Omega, M) \to \mathrm{rv}(\Gamma, N)$ is liftable. Then f is continuous, and for every neoclosed subset C of $\mathrm{rv}(\Gamma, N)$, $f^{-1}(C)$ is neoclosed.*

Proof. Let π be a metric for N, and let $\hat{\pi}$ be the corresponding metric for $\mathrm{rv}(\Omega, N)$. Suppose F lifts f. Let B be a closed subset of $\mathrm{rv}(\Gamma, N)$, and let $x \notin f^{-1}(B)$. Then $f(x) \notin B$ and $\varepsilon = \hat{\pi}(f(x), B) > 0$. Let X lift x. Then whenever $Y \approx X$, $F(Y) \approx F(X)$ and $\bar{\pi}(F(X), F(Y)) \leq \varepsilon/2$. By overspill there is a real $\delta > 0$ such that whenever $\bar{\rho}(X, Y) \leq \delta$, $\bar{\pi}(F(X), F(Y)) \leq \varepsilon/2$. We see from Proposition 2.2 that whenever $\hat{\rho}(x, y) \leq \delta/2$, $\hat{\pi}(f(x), f(y)) \leq \varepsilon/2$ and $\pi(f(y), B) \geq \varepsilon/2$, so $y \notin f^{-1}(B)$. Therefore $f^{-1}(B)$ is closed, and hence f is continuous.

Now let C be a neoclosed subset of $\mathrm{rv}(\Gamma, N)$ and let A_n, $n \in \mathbb{N}$, be a decreasing chain of internal subsets of $\mathrm{RV}(\Gamma, {}^*N)$ such that $C = \mathrm{st}(\bigcap_n A_n)$. Suppose X lifts x. Then $F(X)$ lifts $f(x)$. By Proposition 2.8,

$$x \in f^{-1}(C) \quad \text{iff} \quad f(x) \in C \quad \text{iff} \quad F(X) \in \bigcap_n (A_n)^{1/n} \quad \text{iff} \quad X \in \bigcap_n F^{-1}((A_n)^{1/n}).$$

Therefore $f^{-1}(C) = \mathrm{st}(\bigcap_n f^{-1}((A_n)^{1/n}))$ is neoclosed. □

Proposition 4.4. *A set $C \subseteq \mathrm{rv}(\Omega, M)$ is neoclosed if and only if there is a countable sequence of liftable functions $f_n : \mathrm{rv}(\Omega, M) \to [0, 1]$ such that C is the intersection of the zero sets of f_n, $C = \bigcap_n f_n^{-1}(\{0\})$.*

Proof. By 3.4 and and 4.2, a countable intersection of zero sets of liftable functions is neoclosed.

For the converse, suppose $C = \mathrm{st}(\bigcap_n A_n)$ where A_n is a decreasing chain of internal sets. For each n, define $F_n : \mathrm{RV}(\Omega, {}^*M) \to {}^*[0, 1]$ by

$$F_n(X) = \bar{\rho}(X, (A_n)^{1/n}).$$

Each F_n is internal, and by the triangle inequality, if $X \approx Z$ then $F_n(X) \approx F_n(Z)$. Therefore F_n lifts a liftable function $f_n : \mathrm{rv}(\Omega, M) \to [0, 1]$. We show that $x \in C$ if and only if $f_n(x) = 0$ for all $n \in \mathbb{N}$.

Suppose $x \in C$ and let X lift x. Then $X \in \bigcap_n (A_n)^{1/n}$, so $F_n(X) = 0$ for all n, and $f_n(x) = 0$ for all n.

Now suppose $f_n(x) = 0$ for all n and let X lift x. Then $F_n(X) \approx 0$ for all n, and hence for each n there exists $Y \in A_n$ such that $\bar{\rho}(X, Y) \leqslant 2/n$. By ω_1-saturation there exists $Y \in \bigcap_n A_n$ such that $Y \approx X$. Then Y lifts x, and since $C = \mathrm{st}(\bigcap_n A_n)$ we have $x \in C$. We have shown that C is the intersection of the zero sets of the liftable functions f_n. □

Proposition 4.5. *If $C \subseteq \mathrm{rv}(\Omega, M)$ is neocompact (or tight) and $f : \mathrm{rv}(\Omega, M) \to \mathrm{rv}(\Gamma, N)$ is liftable, then $f(C)$ is neocompact (or tight).*

Proof. Let F lift f. By Corollary 3.5, $C = \mathrm{st}(A)$ for some Π_1^0 set $A \subseteq \mathrm{ns}(\Omega, {}^*M)$. It follows that $f(C) = \mathrm{st}(F(A))$. Let $A = \bigcap_n A_n$ where A_n is a decreasing chain of internal sets. By ω_1-saturation, $F(A) = \bigcap_n F(A_n)$, so $F(A)$ is a Π_1^0 set. If $X \in A$, then X lifts some $x \in C$, so $F(X)$ lifts $f(x)$. Therefore $F(A) \subseteq \mathrm{ns}(\Omega, {}^*M)$. We conclude that $f(C)$ is neocompact. □

Proposition 4.6. *Compositions of liftable functions are liftable. That is, if $f : \mathrm{rv}(\Omega, M) \to \mathrm{rv}(\Lambda, N)$ is liftable and $g : \mathrm{rv}(\Lambda, N) \to \mathrm{rv}(\Gamma, L)$ is liftable, then $g \circ f$ is liftable.*

Proof. If F lifts f and G lifts g, then $G \circ F$ lifts $g \circ f$. □

The notion of a liftable function of finitely many variables is defined in the natural way, and all of the results in this section readily extend to that case.

The following example shows that Proposition 4.5 does not hold for arbitrary continuous functions f.

Example. We build on Example 3.9. Let x_n, $n \in \mathbb{N}$, be a countable sequence of elements of $\mathrm{rv}(\Omega, \{0, 1\})$ such that $P(x_m \triangle x_n) \geqslant 1/2$ whenever $m \neq n$. Then a point $y \in \mathrm{rv}(\Omega, \{0, 1\})$ can be strictly within 1/4 of at most one x_n. For each n, let $f_n(y) = 8n \cdot \max(0, 1/8 - P(x_n \triangle y))$, that is, $f_n : \mathrm{rv}(\Omega, \{0, 1\}) \to \mathbb{R}$ is the function whose graph is a cone with apex of height n at x_n and base the ball of radius 1/8 about x_n. Let $f : \mathrm{rv}(\Omega, \{0, 1\}) \to \mathbb{R}$ be the function $f = \sum_{n=1}^{\infty} f_n$. Each f_n is continuous and each point has a neighborhood on which at most one f_n is nonzero, so f is continuous. Thus:

(a) f is a continuous function whose domain $\mathrm{rv}(\Omega, \{0, 1\})$ is tight but whose range $\mathbb{R}$ is not tight.

(b) By Proposition 4.5, f is not liftable.

5. Forcing and approximations

In this section we introduce a strong language for expressing properties of random variables and a concept of forcing, and prove the main theorem that a statement is true if and only if it is forced. At the end of the section we show that this reduces any sentence in the language to a property of sequences of simple random variables.

Definition of the language $\mathscr{L}$. Given an ω_1-saturated nonstandard universe, a language $\mathscr{L}$ is formed in the following way. $\mathscr{L}$ has infinitely many individual variables v_k. Each variable v_i has a *sort space* $\mathrm{rv}(\Omega_i, M_i)$. We define the sort space of a tuple of variables $\boldsymbol{u} = \langle u_1, \ldots, u_m \rangle$ to be the product of the sort spaces of the u_i.

By a *term* of $\mathscr{L}$ we mean an expression of the form $f(\boldsymbol{u})$ where f is a liftable function from the sort space of $\boldsymbol{u}$ into a space $\mathrm{rv}(\Gamma, M)$. The latter space is called the sort space of the term $f(\boldsymbol{u})$. The *atomic formulas of $\mathscr{L}$* are equations between terms of $\mathscr{L}$ of the same sort. Formulas of $\mathscr{L}$ are formed by starting with the atomic formulas and repeatedly applying negation, finite or countable conjunctions, and existential quantifiers over the variables.

To sum up, $\mathscr{L}$ is a many sorted $L_{\omega_1\omega}$ language which has a sort for each space $\mathrm{rv}(\Omega, M)$, a function symbol for each liftable function, and an equality predicate between terms of each sort. We shall only deal with the intended model for $\mathscr{L}$, where each variable ranges over its sort space and each liftable function is interpreted by itself. Since compositions of liftable functions are again liftable, there is no need to close the set of terms of $\mathscr{L}$ under liftable functions.

We introduce disjunctions and universal quantifiers as abbreviations in the usual way. We shall use various self-explanatory abbreviations for atomic formulas when convenient. For example, if f and g are liftable functions from $\mathrm{rv}(\Omega, L)$ into $\mathrm{rv}(\Gamma, \mathbb{R})$, we shall use $f(u) \geqslant g(u)$ as an abbreviation for the atomic formula $\min(f(u), g(u)) = g(u)$. We may treat a variable alone as a term of $\mathscr{L}$ because the identity function is liftable. By introducing dummy variables, any atomic formula may be written in the form $f(\boldsymbol{u}) = g(\boldsymbol{u})$.

Definition of truth in $\mathscr{L}$. Given an atomic formula $f(\boldsymbol{u})=g(\boldsymbol{u})$ of $\mathscr{L}$ where $f(\boldsymbol{u})$ and $g(\boldsymbol{u})$ are terms of sort $\mathrm{rv}(\Gamma, N)$, and a tuple of random variables $\boldsymbol{a}$ of the same sort as $\boldsymbol{u}$, we say that $f(\boldsymbol{a})=g(\boldsymbol{a})$ is *true* iff $f(\boldsymbol{a})(\gamma)=g(\boldsymbol{a})(\gamma)$ for almost all $\gamma\in\Gamma$. For a formula $\phi(\boldsymbol{u})$ in the language $\mathscr{L}$, the truth value of $\phi(\boldsymbol{a})$ is now defined in the usual way by induction on the complexity of ϕ, with the quantifiers on bound variables ranging over the corresponding sort spaces.

Definition of forcing. A *condition* is an infinite subset p of $\mathbb{N}$. In a metric space M with metric ρ, we say that $\lim_{n\in p} x_n = a$ iff

$$(\forall k\in\mathbb{N})(\exists m\in\mathbb{N})(\forall n\in\mathbb{N})[\text{if } n\in p\wedge n\geqslant m \text{ then } \rho(x_n, a)\leqslant 1/k].$$

The *forcing relation* $p\Vdash\phi(\langle\boldsymbol{x}\rangle)$, where p is a condition, $\phi(\boldsymbol{u})$ is a formula of $\mathscr{L}$ with an m-tuple of free variables $\boldsymbol{u}$, and $\langle\boldsymbol{x}\rangle$ is an m-tuple of tight sequences of the same sort as $\boldsymbol{u}$, is defined by induction on the complexity of ϕ as follows.

(a) If ϕ is an atomic formula $f(\boldsymbol{u})=g(\boldsymbol{u})$ and $f(\boldsymbol{u})$, $g(\boldsymbol{u})$ have sort space $\mathrm{rv}(\Gamma, M)$, then

$$p\Vdash f(\langle\boldsymbol{x}\rangle)=g(\langle\boldsymbol{x}\rangle) \quad\text{iff}\quad \lim_{n\in p}\hat{\rho}(f(\boldsymbol{x}_n), g(\boldsymbol{x}_n))=0.$$

(b) $p\Vdash\neg\phi$ iff there is no condition $q\subseteq p$ such that $q\Vdash\phi$.

(c) $p\Vdash\bigwedge_n\phi_n$ iff $p\Vdash\phi_n$ for all $n\in\mathbb{N}$.

(d) $p\Vdash\exists v\,\phi(\langle\boldsymbol{x}\rangle, v)$ iff for every $q\subseteq p$ there exists $r\subseteq q$ such that $r\Vdash\phi(\langle\boldsymbol{x}\rangle,\langle y\rangle)$ for some tight sequence $\langle y\rangle$ of the same sort as v.

We shall use the word 'formula' for either a formula $\phi(\boldsymbol{u})$ of the language $\mathscr{L}$, or an expression $\phi(\langle\boldsymbol{x}\rangle)$ where $\phi(\boldsymbol{u})$ is a formula of $\mathscr{L}$ and $\langle\boldsymbol{x}\rangle$ is a tight sequence of the same sort as $\boldsymbol{u}$. Thus $\langle\boldsymbol{x}\rangle=\langle\boldsymbol{x}_n : n\in\mathbb{N}\rangle$.

Lemma 5.1. *Let C be a neoclosed subset of* $\mathrm{rv}(\Omega, M)$. *If $x_n\in C$ for all sufficiently large $n\in p$, then $p\Vdash\langle x\rangle\in C$.*

Proof. By Proposition 4.4, C is the intersection of the zero sets of a countable family of liftable functions $f_m:\mathrm{rv}(\Omega, M)\to[0,1]$, $m\in\mathbb{N}$. We take the expression $u\in C$ to be an abbreviation for the formula $\bigwedge_m f_m(u)=0$. It follows from the definition of forcing that $p\Vdash\bigwedge_m f_m(\langle x\rangle)=0$. $\square$

Lemma 5.2. *The following hold for all conditions and formulas.*

(i) *If $p\Vdash\phi$ and $q\subseteq p$, then $q\Vdash\phi$.*

(ii) *$p\Vdash\neg\neg\phi$ if and only if $(\forall q\subseteq p)(\exists r\subseteq q)\ r\Vdash\phi$.*

(iii) *If ϕ is atomic, then $p\Vdash\phi$ if and only if $p\Vdash\neg\neg\phi$.*

(iv) *$p\Vdash\neg\phi$ if and only if $p\Vdash\neg\neg\neg\phi$.*

(v) *$p\Vdash\bigvee_k\phi_k$ if and only if $(\forall q\subseteq p)(\exists r\subseteq q)(\exists k)\ r\Vdash\phi_k$.*

(vi) $p \Vdash \forall v\, \phi(\langle x\rangle, v)$ *if and only if* $p \Vdash \neg\neg\phi(\langle x\rangle, \langle y\rangle)$ *for every tight sequence* $\langle y\rangle$ *of the same sort as* v.

(vii) *If* $p \,\triangle\, q$ *is finite, then* $p \Vdash \phi$ *if and only if* $q \Vdash \phi$.

Proof. We prove the nontrivial direction of (iii). Assume that ϕ is the atomic formula $f(\langle x\rangle) = g(\langle x\rangle)$ and not $p \Vdash \phi$. Then there is an infinite $q \subseteq p$ and a real $b > 0$ such that $\hat{\rho}(f(x_n), g(x_n)) \geqslant b$ for all $n \in q$. Therefore there is no $r \subseteq q$ such that $r \Vdash \phi$, so not $p \Vdash \neg\neg\phi$. The other parts of the lemma are routine. □

We now come to our main theorem, which will allow us to reduce statements about continuous time processes to statements about discrete time processes. It shows that a sentence in the language $\mathscr{L}$ is true if and only if it is forced by some, or every, condition. This result will be applied in a variety of settings in Section 8. In these applications, a sentence ϕ in the language $\mathscr{L}$ is proved by showing that ϕ is forced. In many cases, the property that ϕ is forced will be an easy fact about discrete time, while the statement ϕ itself will be an apparently more difficult fact about continuous time.

Theorem 5.3. *Let* p *be a condition. Suppose* $\langle x\rangle$ *is a tight sequence and* $\lim_{n\in p} x_n = a$. *Then for all formulas* $\phi(\langle x\rangle)$, $p \Vdash \phi(\langle x\rangle)$ *if and only if* $\phi(a)$ *is true. In particular, if* ϕ *is a sentence, then* $p \Vdash \phi$ *if and only if* ϕ *is true.*

Remark. The 'deterministic' case of Theorem 5.3, where the sort space of each variable is a Polish space M instead of a space of random variables $\mathrm{rv}(\Omega, M)$, may be proved by a very easy induction which we leave as an exercise. The only step of this induction which breaks down in the general case is the existential quantifier step in the direction from $p \Vdash \exists v\, \phi(\langle x\rangle, v)$ to $\exists v\, \phi(a, v)$. In the deterministic case this step depends on the fact that every tight sequence in M has a convergent subsequence (by 1.1). It goes as follows: Suppose $\lim_{n\in p} x_n = a$ and $p \Vdash \exists v\, \phi(\langle x\rangle, v)$. Then $q \Vdash \phi(\langle x\rangle, \langle y\rangle)$ for some condition $q \subseteq p$ and some tight sequence $\langle y\rangle$, and $\lim_{n\in r} y_n = b$ exists for some $r \subseteq q$. Then $r \Vdash \phi(\langle x\rangle, \langle y\rangle)$, so $\phi(a, b)$ holds by inductive hypothesis, whence $\exists v\, \phi(a, v)$ holds.

This argument fails in the probabilistic case because a tight sequence in $\mathrm{rv}(\Omega, M)$ will ordinarily not have a convergent subsequence in the sense of the metric space. Instead of proving Theorem 5.3 directly, we give an inductive proof of a stronger result (Theorem 5.9) where the tight sequence $\langle x\rangle$ does not necessarily converge. We first prove a series of lemmas.

Definition. Given a condition p, we say that a set A of hyperintegers *contains all sufficiently small infinite* $J \in {}^*p$ iff there is an infinite $K \in {}^*\mathbb{N}$ such that $J \in A$ for all infinite $J \in {}^*p$ with $J \leqslant K$. We say that A *contains arbitrarily small infinite* $J \in {}^*p$ iff it is not the case that ${}^*\mathbb{N}\setminus A$ contains all sufficiently small infinite $J \in {}^*p$ (that is, for each infinite $K \in {}^*\mathbb{N}$ there exists an infinite $J \in {}^*p$ such that $J \leqslant K$ and $J \in A$).

The next lemma gives a criterion for an atomic formula to be forced. Later on, we shall extend this criterion to arbitrary formulas.

Lemma 5.4. *Consider a condition p and an atomic formula $f(\langle \boldsymbol{x} \rangle) = g(\langle \boldsymbol{x} \rangle)$, and let $\langle \boldsymbol{X}_n : n \in {}^*\mathbb{N} \rangle$ be a lifting of $\langle \boldsymbol{x} \rangle$. Then $p \Vdash f(\langle \boldsymbol{x} \rangle) = g(\langle \boldsymbol{x} \rangle)$ if and only if $f(\mathrm{st}(\boldsymbol{X}_J)) = g(\mathrm{st}(\boldsymbol{X}_J))$ for all sufficiently small infinite $J \in {}^*p$.*

Proof. Let F lift f and G lift g. By Theorem 3.2, since $\langle \boldsymbol{x} \rangle$ is tight, $\boldsymbol{X}_J$ is near-standard for all sufficiently small infinite J. Since F lifts f, $\mathrm{st}(F(\boldsymbol{X}_J)) = f(\mathrm{st}(\boldsymbol{X}_J))$ for all sufficiently small infinite J, and similarly for G.

Suppose $p \Vdash f(\langle \boldsymbol{x} \rangle) = g(\langle \boldsymbol{x} \rangle)$. Then $\lim_{n \in p} \hat{\rho}(f(\boldsymbol{x}_n), g(\boldsymbol{x}_n)) = 0$. By overspill, for each $k \in \mathbb{N}$, for all sufficiently small infinite $J \in {}^*p$ we have $\bar{\rho}(F(\boldsymbol{X}_J), G(\boldsymbol{X}_J)) \leq 1/k$. By ω_1-saturation, for all sufficiently small infinite $J \in {}^*p$ we have $F(\boldsymbol{X}_J) \approx G(\boldsymbol{X}_J)$. It follows that

$$f(\mathrm{st}(\boldsymbol{X}_J)) = \mathrm{st}(F(\boldsymbol{X}_J)) = \mathrm{st}(G(\boldsymbol{X}_J)) = g(\mathrm{st}(\boldsymbol{X}_J))$$

for all sufficiently small infinite $J \in {}^*p$.

Now suppose $f(\mathrm{st}(\boldsymbol{X}_J)) = g(\mathrm{st}(\boldsymbol{X}_J))$ for all sufficiently small infinite $J \in {}^*p$. Then $\bar{\rho}(F(\boldsymbol{X}_J), G(\boldsymbol{X}_J)) \approx 0$ for all such J. By overspill, for each $k \in \mathbb{N}$, $\bar{\rho}(F(\boldsymbol{X}_n), G(\boldsymbol{X}_n)) \leq 1/K$ for all sufficiently large $n \in p$. Since F and G lift f and g, for each $k \in \mathbb{N}$, we have $\hat{\rho}(f(\boldsymbol{x}_n), g(\boldsymbol{x}_n)) \leq 2/k$ for all sufficiently large $n \in p$. Therefore $\lim_{n \in p} \hat{\rho}(f(\boldsymbol{x}_n), g(\boldsymbol{x}_n)) = 0$, whence $p \Vdash f(\langle \boldsymbol{x} \rangle) = g(\langle \boldsymbol{x} \rangle)$. □

To extend the above lemma to arbitrary formulas, we shall use the notion of a countably determined set, which was introduced by Henson [12]. A subset A of an internal set I is said to be *countably determined* iff there is a countable collection $\{B_n : n \in \mathbb{N}\}$ of internal subsets of I such that A is a (possibly infinite) Boolean combination of the B_n's. The collection of countably determined sets is closed under countable unions, complements, and images and preimages under internal functions.

Lemma 5.5. *For each formula $\phi(\boldsymbol{u})$ in the language $\mathscr{L}$, the set $T(\phi)$ of all internal tuples $\boldsymbol{X}$ such that $\boldsymbol{X}$ is near-standard, $\boldsymbol{X}$ has the same sort as $\boldsymbol{u}$, and $\phi(\mathrm{st}(\boldsymbol{X}))$ is true, is countably determined.*

Proof. By Proposition 2.6, the set of all internal near-standard tuples $\boldsymbol{X}$ is countably determined. We argue by induction on the complexity of ϕ. If ϕ is an atomic formula $f(\boldsymbol{u}) = g(\boldsymbol{u})$, and F, G lift f, g, then $T(\phi)$ is the set of all near-standard internal $\boldsymbol{X}$ which belong to the intersection of the countable chain of internal sets

$$B_n = \{\boldsymbol{X} : \bar{\rho}(F(\boldsymbol{X}), G(\boldsymbol{X})) \leq 1/n\},$$

and thus $T(\phi)$ is countably determined. The induction steps for negation, countable disjunction, and existential quantifier follows from the fact that the countably determined sets are closed under complements, countable unions, and projections. □

Corollary 5.6. *Let $\phi(\boldsymbol{u}, v)$ be a formula of $\mathscr{L}$ and let $\langle \boldsymbol{X}_J : j \in {}^*\mathbb{N}\rangle$ be an internal sequence of tuples of the same sort as $\boldsymbol{u}$. Then the set*

$$\{\langle J, Y\rangle \in {}^*\mathbb{N} \times \mathrm{RV}(\Omega, {}^*M) : \langle \boldsymbol{X}_J, Y\rangle \in T(\phi)\}$$

is countably determined.

Proof. The set in question is the preimage of $T(\phi)$ under the internal function $\langle J, Y\rangle \mapsto \langle \boldsymbol{X}_j, Y\rangle$. □

Lemma 5.7. *Let A be a countably determined subset of an internal set I, let $F: I \to {}^*\mathbb{N}$ be an internal function, and let p be a condition such that $F(A)$ contains arbitrarily small infinite $J \in {}^*p$. Then A has a Π_1^0 subset B such that $F(B)$ contains arbitrarily small infinite $J \in {}^*p$.*

Proof. Let A be determined by the countable collection $\{B_n : n \in \mathbb{N}\}$ of internal subsets of i. For each $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ let

$$B_{n,X} = B_n \quad \text{if } n \in X, \qquad B_{n,X} = I \setminus B_n \quad \text{if } n \notin X.$$

Let the type $t(X)$ of X be the Π_1^0 set

$$t(X) = \bigcap_{n \in \mathbb{N}} B_{n,X}.$$

Then for some set $S \subseteq \mathscr{P}(\mathbb{N})$,

$$A = \bigcup_{X \in S} t(X).$$

Thus if $X \in S$ then $t(X) \subseteq A$, and otherwise $t(X) \cap A = \emptyset$. To prove the lemma, it suffices to show that for some $X \in S$, $F(t(X))$ contains arbitrarily small infinite $J \in {}^*p$. Suppose not. Then for each $X \in S$ there exists an infinite $K \in {}^*\mathbb{N}$ such that $F(t(X))$ contains no infinite $J \in {}^*p$ with $J \leqslant K$. By ω_1-saturation, for each $X \in S$ there exists an infinite $K \in {}^*\mathbb{N}$ and an $m \in \mathbb{N}$ such that $F(\bigcap_{n \leqslant m} B_{n,X})$ contains no infinite $J \in {}^*p$ with $J \leqslant K$. Since there are only countably many different internal sets of the form $\bigcap_{n \leqslant m} B_{n,X}$, it follows by ω_1-saturation that there exists an infinite $K \in {}^*\mathbb{N}$ such that for each $X \in S$, $F(t(X))$ contains no infinite $J \in {}^*p$ with $J \leqslant K$. But then $F(A)$ contains no infinite $J \in {}^*p$ with $J \leqslant K$, a contradiction. This completes the proof. □

Lemma 5.8. *Let A be a countably determined subset of ${}^*\mathbb{N}$ and let p be a condition. Either A contains all sufficiently small infinite elements of ${}^*p$, or there is a condition $q \subseteq p$ such that ${}^*\mathbb{N} \setminus A$ contains all sufficiently small infinite elements of*

$^*q$. If A contains arbitrarily small infinite elements of $^*p$, then there is a condition $q \subseteq p$ such that A contains all sufficiently small infinite elements of $^*q$.

Proof. Suppose A does not contain all sufficiently small infinite $J \in {}^*p$ and let $B = {}^*\mathbb{N} \setminus A$. Then B contains arbitrarily small infinite $J \in {}^*p$. By Lemma 5.7, there is a Π_1^0 set $C \subseteq B$ which contains arbitrarily small infinite $J \in {}^*p$. Let $C = \bigcap_n C_n$ where C_n is a decreasing chain of internal sets. By overspill, each C_n contains arbitrarily large elements of p. Choose a strictly increasing sequence a_n in $\mathbb{N}$ so that $a_n \in p \cap C_n$. Let $q = \{a_n : n \in \mathbb{N}\}$. Then q is a condition and $q \subseteq p$. For each $n \in \mathbb{N}$, we have $a_j \in C_n$ for all finite $j \geqslant n$. By overspill, for each n there is an infinite K such that $^*a_J \in C_n$ for all infinite $J \leqslant K$. By ω_1-saturation there is an infinite K such that $^*a_J \in C$ for all infinite $J \leqslant K$. Since the sequence a_n is increasing and has range q, it follows that C and hence B contains all sufficiently small infinite $J \in {}^*q$.

Now suppose A contains arbitrarily small infinite $J \in {}^*p$ and let $B = {}^*\mathbb{N} \setminus A$. Then B does not contain all sufficiently small infinite $J \in {}^*p$. By the first paragraph, there is a condition $q \subseteq p$ such that ${}^*\mathbb{N} \setminus B = A$ contains all sufficiently small infinite $J \in {}^*q$. $\square$

We now prove the analogue of Lemma 5.4 for arbitrary formulas of $\mathscr{L}$.

Theorem 5.9. *Consider a condition p and formula $\phi(\langle \boldsymbol{x} \rangle)$ of $\mathscr{L}$, and let $\langle \boldsymbol{X}_J : J \in {}^*\mathbb{N} \rangle$ be a lifting of $\langle \boldsymbol{x} \rangle$. Then $p \Vdash \phi(\langle \boldsymbol{x} \rangle)$ if and only if $\phi(\mathrm{st}(\boldsymbol{X}_J))$ holds for all sufficiently small infinite $J \in {}^*p$.*

Proof. We argue by induction on the complexity of ϕ. By Lemma 5.4, the result holds when ϕ is an atomic formula.

For the negation step, we assume the result holds for ϕ and prove it for $\neg\phi$. Suppose that $\neg\phi(\mathrm{st}(\boldsymbol{X}_J))$ holds for all sufficiently small infinite $J \in {}^*p$. Then there is no condition $q \subseteq p$ such that $\phi(\mathrm{st}(\boldsymbol{X}_J))$ holds for all sufficiently small infinite $J \in {}^*q$. Therefore by the induction hypothesis, no $q \subseteq p$ forces $\phi(\langle \boldsymbol{x} \rangle)$, and hence $p \Vdash \neg\phi(\langle \boldsymbol{x} \rangle)$. Now suppose that it is not the case that $\neg\phi(\mathrm{st}(\boldsymbol{X}_J))$ holds for all sufficiently small infinite $J \in {}^*p$. By Corollary 5.6 and Lemma 5.8, there is a condition $q \subseteq p$ such that $\phi(\mathrm{st}(\boldsymbol{X}_J))$ holds for all sufficiently small infinite $J \in {}^*q$. Then by inductive hypothesis, $q \Vdash \phi(\langle \boldsymbol{x} \rangle)$, and hence p does not force $\neg\phi(\langle \boldsymbol{x} \rangle)$. This proves the result for $\neg\phi$.

We now assume that the result holds for each ϕ_n, $n \in \mathbb{N}$, and prove that the result holds for $\phi = \bigwedge_n \phi_n$. Suppose $p \Vdash \phi(\langle \boldsymbol{x} \rangle)$. Then $p \Vdash \phi_n(\langle \boldsymbol{x} \rangle)$ for all n. By the induction hypothesis, for each n, $\phi_n(\mathrm{st}(\boldsymbol{X}_J))$ holds for all sufficiently small infinite $J \in {}^*p$. By ω_1-saturation, $\phi(\mathrm{st}(\boldsymbol{X}_J))$ holds for all sufficiently small infinite $J \in {}^*p$. If $\phi(\mathrm{st}(\boldsymbol{X}_J))$ holds for all sufficiently small infinite $J \in {}^*p$, then so does each $\phi_n(\mathrm{st}(\boldsymbol{X}_J))$, so by the induction hypothesis, p forces each $\phi_n(\langle \boldsymbol{x} \rangle)$ and p forces $\phi(\langle \boldsymbol{x} \rangle)$. Thus the result holds for ϕ.

Finally, we assume that the result holds for $\phi(\boldsymbol{u}, v)$ and prove the result for $\exists v\, \phi(\boldsymbol{u}, v)$.

Suppose first that $p \Vdash \exists v\, \phi(\langle \boldsymbol{x} \rangle, v)$. Let A be the set of all $J \in {}^*\mathbb{N}$ such that $\exists v\, \phi(\mathrm{st}(\boldsymbol{X}_J), v)$ holds. A is countably determined by Corollary 5.6. We wish to show that A contains all sufficiently small infinite $J \in {}^*\mathbb{N}$. Suppose not. By Lemma 5.8, there is a condition $q \subseteq p$ such that ${}^*\mathbb{N} \backslash A$ contains all sufficiently small infinite $J \in {}^*q$. Since p forces $\exists v\, \phi(\langle \boldsymbol{x} \rangle, v)$, there is a condition $r \subseteq q$ and a tight sequence $\langle y \rangle$ with the same sort as v such that $r \Vdash \phi(\langle \boldsymbol{x} \rangle, \langle y \rangle)$. Let $\langle Y_n : n \in {}^*\mathbb{N} \rangle$ be a lifting of $\langle y \rangle$. By induction hypothesis, $\phi(\mathrm{st}(\boldsymbol{X}_J), \mathrm{st}(Y_J))$ holds for all sufficently small infinite $J \in {}^*r$, and so does $\exists v\, \phi(\mathrm{st}(\boldsymbol{X}_J), v)$. But then A contains all sufficiently small infinite $J \in {}^*r$, contradicting the fact that ${}^*\mathbb{N} \backslash A$ contains all sufficiently small infinite $J \in {}^*q$.

Now suppose $\exists v\, \phi(\mathrm{st}(\boldsymbol{X}_J), v)$ holds for all sufficiently small infinite $J \in {}^*p$. We shall prove that $p \Vdash \exists v\, \phi(\langle \boldsymbol{x} \rangle, v)$. Let $q \subseteq p$ and let

$$A = \{\langle J, Y \rangle : J \in {}^*p,\ Y \text{ is near-standard, and } \phi(\mathrm{st}(\boldsymbol{X}_J), \mathrm{st}(Y)) \text{ holds}\}.$$

By Lemma 5.5, A is countably determined. Let F and G be the internal functions $F(J, Y) = J$, $G(J, Y) = Y$. Then $F(A)$ contains all sufficiently small infinite $J \in {}^*p$, and hence contains all sufficiently small infinite $J \in {}^*q$. By Lemma 5.7, a has a Π_1^0 subset B such that $F(B)$ contains arbitrarily small infinite $J \in {}^*q$. Then by Lemma 5.8, there is a condition $r \subseteq q$ such that $F(B)$ contains all sufficiently small infinite $J \in {}^*r$. By ω_1-saturation, the set

$$C = \{\langle J, Y \rangle : \bar{\rho}(Y, W) \approx 0 \text{ for some } W \text{ with } \langle J, W \rangle \in B\}$$

is also a Π_1^0 set. Since the relation $\langle J, Y \rangle \in A$ depends on $\mathrm{st}(Y)$ rather than on Y, we have $B \subseteq C \subseteq A$. Let $C = \bigcap_n C_n$ where C_n, $n \in \mathbb{N}$, is a decreasing chain of internal sets. By ω_1-saturation, $G(C) = \bigcap_n G(C_n)$, so $G(C)$ is a Π_1^0 set. We have $G(C) \subseteq G(A) \subseteq \mathrm{ns}(\Omega, {}^*M)$, so by Corollary 3.5, $\mathrm{st}(G(C))$ is neocompact (and hence tight). By Theorem 3.1, there is an internal set $D \subseteq \mathrm{ns}(\Omega, {}^*M)$ such that $\mathrm{st}(G(C)) \subseteq \mathrm{st}(D)$. Let $E = C \cap ({}^*\mathbb{N} \times D)$ and $E_n = C_n \cap ({}^*\mathbb{N} \times D)$. Then $E = \bigcap_n E_n$, $F(E) = F(C)$, $\mathrm{st}(G(E)) = \mathrm{st}(G(C))$, and for each n, $G(E_n) \subseteq \mathrm{ns}(\Omega, {}^*M)$. By overspill, for each n there is a natural number a_n such that $F(E_n)$ contains all finite $j \geqslant a_n$ such that $j \in r$. Since the sets E_n are decreasing, the sequence a_n is increasing. We may therefore choose Y_j for each $j \in r$ such that whenever $a_n \leqslant j$, $\langle j, Y_j \rangle \in E_n$. Let z be near-standard, and for $j \notin r$ let $Y_j = Z$. Since each element of $G(E_n)$ is near-standard, Y_j is near-standard for each $j \in \mathbb{N}$. By ω_1-saturation we may extend the sequence Y_j, $j \in \mathbb{N}$ to an internal sequence $\langle Y_J : J \in {}^*\mathbb{N} \rangle$. Using ω_1-saturation again, for all sufficiently small infinite J, we have $\langle J, Y_J \rangle \in E$ if $J \in {}^*r$, and $Y_J = Z$ if $J \notin {}^*r$, and hence Y_J is near-standard. Let $\langle y \rangle$ be the sequence defined by $y_n = \mathrm{st}(Y_n)$. By Theorem 3.2, $\langle y \rangle$ is tight. Then for all sufficiently small infinite $J \in {}^*r$, we have $\langle J, Y_J \rangle \in E \subseteq A$, and therefore $\phi(\mathrm{st}(\boldsymbol{X}_J), \mathrm{st}(Y_J))$ holds. By the induction hypothesis, $r \Vdash \phi(\langle \boldsymbol{x} \rangle, \langle y \rangle)$, and therefore $p \Vdash \exists v\, \phi(\langle \boldsymbol{x} \rangle, v)$ as required. This shows that the result holds for $\exists v\, \phi(\boldsymbol{u}, v)$. $\square$

Proof of Theorem 5.3. Let $\langle X_J : J \in {}^*\mathbb{N}\rangle$ be a lifting of $\langle \boldsymbol{x}\rangle$. By Proposition 2.3, $\mathrm{st}(X_J) = \boldsymbol{a}$ for all sufficiently small infinite $J \in {}^*p$. Therefore by Theorem 5.9, $p \Vdash \phi(\langle \boldsymbol{x}\rangle)$ if and only if $\phi(\boldsymbol{a})$ is true. $\square$

Remark. The hardest step in the inductive proof of Theorem 5.9 is the half of the quantifier step whose conclusion is $p \Vdash \exists v\, \psi(\langle \boldsymbol{x}\rangle, v)$. In the sample applications at the end of this paper, Theorem 5.3 is only needed in the special case that if $\phi(\boldsymbol{u})$ is an existential formula of $\mathscr{L}$ and $\langle \boldsymbol{x}\rangle$ approximates $\boldsymbol{a}$, then $p \Vdash \phi(\langle \boldsymbol{x}\rangle)$ implies $\phi(\boldsymbol{a})$. The proof of this special case is easier than the proof of the full result because it does not use the hard quantifier step.

We now turn to approximating sequences.

Definition. Two sequences $\langle x\rangle$ and $\langle y\rangle$ in $\mathrm{rv}(\Omega, M)$ are said to *approximate* each other iff $\lim_{n\to\infty} \hat{\rho}(x_n, y_n) = 0$. The sequence $\langle y\rangle$ *approximates* an element $x \in \mathrm{rv}(\Omega, M)$ iff $\lim_{n\to\infty} \hat{\rho}(x, y_n) = 0$. Given an increasing chain M_n, $n \in \mathbb{N}$ of subsets of M, we shall say that $\langle x\rangle$ is a *sequence* through M_n iff $x_n \in \mathrm{rv}(\Omega, M_n)$ for each $n \in \mathbb{N}$.

Proposition 5.10. *Suppose two sequences $\langle x\rangle$ and $\langle y\rangle$ in $\mathrm{rv}(\Omega, M)$ approximate each other and $\langle x\rangle$ is tight. Then $\langle y\rangle$ is tight. Moreover, if $\langle X_J : J \in {}^*\mathbb{N}\rangle$ and $\langle Y_J : J \in {}^*\mathbb{N}\rangle$ are liftings of $\langle x\rangle$ and $\langle y\rangle$, then $X_J \approx Y_J$ for all sufficiently small infinite J.*

Proof. By Theorem 3.2, X_J is near-standard for all sufficiently small infinite J. By overspill, for each $m \in \mathbb{N}$, $\hat{\rho}(X_J, Y_J) \leqslant 1/m$ for all sufficiently small infinite J. Thus by 1.9, $X_J \approx Y_J$ for all sufficiently small infinite J. Then Y_J is near-standard for all sufficiently small infinite J, so $\langle y\rangle$ is tight by Theorem 3.2. $\square$

Corollary 5.11. *If $p \Vdash \phi(\langle \boldsymbol{x}\rangle)$ and $\langle \boldsymbol{y}\rangle$ approximates $\langle \boldsymbol{x}\rangle$, then $p \Vdash \phi(\langle \boldsymbol{y}\rangle)$.*

Proof. By Proposition 5.10 and Theorem 5.9. $\square$

The following proposition shows that in many cases, every tight sequence of random variables is approximated by a tight sequence of 'simple' random variables.

Proposition 5.12. *Let M_n, $n \in \mathbb{N}$, be an increasing chain of subsets of M such that $\bigcup_n M_n$ is dense in M. Then every tight sequence in $\mathrm{rv}(\Omega, M)$ is approximated by a tight sequence through M_n.*

Proof. Let $\langle y_n\rangle$ be a tight sequence in $\mathrm{rv}(\Omega, M)$, and for each $m \in \mathbb{N}$ let K_m be a compact subset of M such that for all n, $y_n \in K_m$ with probability $\geqslant 1 - 1/m$. For each m, there is a finite set $C_m \subseteq \bigcup_n M_n$ such that every element of K_m is within $1/m$ of an element of C_m. There is a Borel function $f_m : M \to C_m$ such that for all $a \in K_m$, $\rho(a, f_m(a)) < 1/m$. For each n, let $h(n)$ be the greatest $m \leq n$ such that $C_m \subseteq M_n$. Then $\lim_{n\to\infty} h(n) = \infty$. Define the sequence $\langle x_n \rangle$ by $x_n = f_{h(n)}(y_n)$. Then whenever $h(n) \geq m$, we have $\rho(x_n, y_n) < 1/m$ with probability $\geq 1 - 1/m$, and therefore $\hat{\rho}(x_n, y_n) < 1/m$. It follows that $\langle x \rangle$ approximates $\langle y \rangle$. Moreover, $\langle x \rangle$ is tight by Proposition 5.10, and $x_n \in \mathrm{rv}(\Omega, M_n)$ because $\mathrm{range}(x_n) \subseteq C_{h(n)} \subseteq M_n$. □

Examples. In each of the following, M_n is a chain of sets such that $\bigcup_n M_n$ is dense in a Polish space M, and thus Proposition 5.12 applies.

(a) M_n is the set of all elements of $\mathbb{R}^d$ which are 2^{-n}-lattice points in $[-2^n, 2^n]^d$.

(b) L is a Polish space, and M_n is the set of all elements of $M = \mathrm{rv}([0, 1], L)$ which are step functions with steps at multiples of 2^{-n}.

(c) L is a Polish space, $\mathbb{B}$ is a countable subset of $[0, \infty)$, $\mathbb{B}_n$, $n \in \mathbb{N}$, is a chain of finite sets with $\bigcup_n \mathbb{B}_n = \mathbb{B}$, and M_n is the set of all elements of $M = L^{\mathbb{B}}$ which are step functions with steps in $\mathbb{B}_n$.

(d) L is a linear Polish space, and M_n is the set of all elements of $M = C([0, 1]^d, L)$ which are polygonal functions whose vertices are 2^{-n}-lattice points.

The next corollary shows that forcing can be defined strictly in terms of sequences of 'simple' random variables, that is sequences through some chain M_n. Thus in the preceding examples, forcing is equivalent to a statement about sequences of discrete time processes.

Corollary 5.13. *Let M_n, $n \in \mathbb{N}$ be a chain of sets such that $\bigcup_n M_n$ is dense in a Polish space M. Let v be a variable with sort space $\mathrm{rv}(\Omega, M)$. Then for any condition p and formula $\exists v\, \phi(\langle \boldsymbol{x} \rangle, v)$, we have $p \Vdash \exists v\, \phi(\langle \boldsymbol{x} \rangle, v)$ if and only if for every $q \subseteq p$ there is an $r \subseteq q$ and a tight sequence $\langle y \rangle$ through M_n such that $r \Vdash \phi(\langle \boldsymbol{x} \rangle, \langle y \rangle)$.*

Proof. By Corollary 5.11 and Proposition 5.12. □

6. A gallery of liftable functions

In this section we shall obtain a series of sufficient conditions for a function to be liftable. Since preimages of neoclosed sets under liftable functions are neoclosed, these results will also give sufficient conditions for a set to be neoclosed. These results will enchance the applicability of Theorem 5.3, because any equation between liftable functions can be expressed in the language $\mathscr{L}$.

Proposition 6.1. *Let h be an internal function $h:\Omega\to\Gamma$ such that whenever A has Loeb measure 0 in Γ, $h^{-1}(A)$ has Loeb measure 0 in Ω. Then the function $f:\mathrm{rv}(\Gamma, M)\to\mathrm{rv}(\Omega, M)$ defined by $f(x)(\omega)=x(h\omega)$ is liftable.*

Proof. If $x:\Gamma\to M$ is Loeb measurable, then $x\circ h:\Omega\to M$ is Loeb measurable, because for any Borel set $A\subseteq M$, $x^{-1}(A)$ has the form $B\triangle C$ where $B\in\mathrm{Borel}(\Gamma)$ and C has Loeb measure 0 in Γ, and by hypothesis we have $h^{-1}(x^{-1}(A))=h^{-1}(B)\triangle h^{-1}(C)$ where $h^{-1}(B)\in\mathrm{Borel}(\Omega)$ and $h^{-1}(C)$ has Loeb measure 0 in Ω. If x is equivalent to y, then $x\circ h$ is equivalent to $y\circ h$ because

$$\{\omega\in\Omega: x(h\omega)\neq y(h\omega)\}=h^{-1}\{\gamma\in\Gamma: x(\gamma)\neq y(\gamma)\}.$$

Thus f is well defined and maps $\mathrm{rv}(\Gamma, M)$ into $\mathrm{rv}(\Omega, M)$. Define $F:\mathrm{RV}(\Gamma^*, M)\to\mathrm{RV}(\Omega, {}^*M)$ by $F(X)=X\circ h$. Suppose X lifts x. Let $A=\{\gamma\in\Gamma: \mathrm{st}(X(\gamma))\neq x(\gamma)\}$. Then A has Loeb measure 0, so $h^{-1}(A)$ has Loeb measure 0. We have $h^{-1}(A)=\{\omega\in\Omega: \mathrm{st}(X(h\omega))\neq x(h\omega)\}$. Therefore $X\circ h$ lifts $x\circ h$, and F is a lifting of f. □

Remark. Here are some special cases of 6.1.

(a) The function $f:M\to\mathrm{rv}(\Omega, M)$, where $f(a)$ is the constant function with value a, is liftable. This case arises when h is the trivial function $h:\Omega\to 1$.

(b) The function $f:\mathrm{rv}(\Gamma, M)\to\mathrm{rv}(\Omega\times\Gamma, M)$ is liftable, where $(fx)(\omega,\gamma)=x(\gamma)$. This case arises when h is the projection function $h:\Omega\times\Gamma\to\Gamma$.

(c) For any internal permutation $h:\Omega\to\Omega$, the function $f:\mathrm{rv}(\Omega, M)\to\mathrm{rv}(\Omega, M)$ is liftable where $f(x)=x\circ h$.

Proposition 6.2. *Let $f:\mathrm{rv}(\Gamma, L)\to\mathrm{rv}(\Gamma, M)$ be liftable. Then the function $g:\mathrm{rv}(\Omega\times\Gamma, L)\to\mathrm{rv}(\Omega\times\Gamma, M)$ defined by the rule*

$$g(x(\cdot,\cdot))(\omega,\gamma)=f(x(\omega,\cdot))(\gamma)$$

is liftable.

Proof. Let F lift f. Define $G:\mathrm{RV}(\Omega\times\Gamma, {}^*L)\to\mathrm{RV}(\Omega\times\Gamma, {}^*M)$ by the rule

$$G(X(\cdot,\cdot))(\omega,\gamma)=F(X(\omega,\cdot))(\gamma).$$

We show that G is a lifting of g. Let $X\in\mathrm{RV}(\Omega\times\Gamma, {}^*L)$ lift $x\in\mathrm{rv}(\Omega\times\Gamma, L)$. By the hyperfinite Fubini theorem in [18, Theorem 1.14], $X(\omega,\cdot)$ lifts $x(\omega,\cdot)$. Therefore $F(X(\omega,\cdot))$ lifts $f(x(\omega,\cdot))$ for almost all $\omega\in\Omega$. For each such ω, $F(X(\omega,\cdot))(\gamma)$ lifts $f(x(\omega,\cdot))(\gamma)$ for almost all $\gamma\in\Gamma$. By the hyperfinite Fubini theorem again, we conclude that $G(X(\cdot,\cdot))(\omega,\gamma)$ lifts $g(x(\cdot,\cdot))(\omega,\gamma)$ for almost all $(\omega,\gamma)\in\Omega\times\Gamma$, whence G lifts g. □

Corollary 6.3. *If $f: M \to N$ is continuous, then the function $g: \mathrm{rv}(\Omega, M) \to \mathrm{rv}(\Omega, N)$ defined by $g(x(\cdot))(\omega) = f(x(\omega))$ is liftable.*

Proof. Apply Proposition 6.2 with $\Gamma = 1$; f is liftable by 1.8. □

Proposition 6.4. *Let $[a, b]$ be a closed real interval. The function $g: \mathrm{rv}(\Omega, [a, b]) \to [a, b]$ defined by $g(x) = E[x]$ is liftable.*

Proof. Let X lift x where $x \in \mathrm{rv}(\Omega, [a, b])$. Then X has values in $^*[a, b]$. By Loeb [23], $\bar{E}[X] \approx E[x]$. Therefore the function G defined by $G(X) = \bar{E}[X]$ lifts g.

Here is an alternative proof. The function g may be written as the composition

$$g(x) = I(\mathrm{law}(x))$$

where $I: \mathrm{Meas}([a, b]) \to [a, b]$ is defined by $I(\mu) = \int t \, \mathrm{d}\mu(t)$. I is continuous and thus liftable by 1.8. The function law is liftable by 2.4, so g is also liftable by 4.3. □

Proposition 6.5. *Let $[a, b]$ be a closed real interval, and let $\Omega \times \Gamma$ be the cartesian product of two hyperfinite sets with the internal counting probability measure. Then the function $g: \mathrm{rv}(\Omega \times \Gamma, [a, b]) \to \mathrm{rv}(\Gamma, [a, b])$ defined by*

$$g(x)(\gamma) = E[x(\cdot, \gamma)]$$

is liftable.

Proof. This follows from the proof of Proposition 1.14 in [18], which is a hyperfinite analogue of the Fubini theorem. It is shown there that if $x \in \mathrm{rv}(\Omega \times \Gamma, [a, b])$ then $x(\cdot, \gamma) \in \mathrm{rv}(\Omega, [a, b])$ for almost all γ, that $g(x) \in \mathrm{rv}(\Gamma, [a, b])$, and that g is lifted by the internal function $G: \mathrm{RV}(\Omega \times \Gamma, ^*[a, b]) \to \mathrm{RV}(\Gamma, ^*[a, b])$ defined by

$$G(X)(\gamma) = \bar{E}[X(\cdot, \gamma)]. \quad \square$$

The following sufficient condition for liftability concerns conditional expectations. The conditional expectation of a P-integrable random variable $x \in \mathrm{rv}(\Omega, [a, b])$ with respect to a σ-algebra $\mathcal{B}$ is denoted by $E[x \mid \mathcal{B}]$. We let $\mathbb{1}_A$ denote the characteristic function of a set A. By definition, $E[x \mid \mathcal{B}]$ is the P-almost surely unique random variable y such that for every $A \in \mathcal{B}$, $E[x \cdot \mathbb{1}_A] = E[y \cdot \mathbb{1}_A]$.

Proposition 6.6. *Suppose $[a, b]$ is a closed real interval and $\mathcal{B}$ is a σ-algebra on Ω which is generated by an internal algebra $\mathcal{A}$ of internal subsets of Ω. Then the function $g: \mathrm{rv}(\Omega, [a, b]) \to \mathrm{rv}(\Omega, [a, b])$ defined by*

$$g(x) = E[x \mid \mathcal{B}]$$

is liftable.

Proof. Let $G:\mathrm{RV}(\Omega, {}^*[a, b]) \to \mathrm{RV}(\Omega, {}^*[a, b])$ be defined by $G(X) = \bar{E}[X \mid \mathcal{A}]$. We show that G is a lifting of g. Let $x \in \mathrm{rv}(\Omega, [a, b])$ and let X lift x. $G(X)$ lifts y where $y = \mathrm{st}(\bar{E}[X \mid \mathcal{A}])$. The function y is $\mathcal{B}$-measurable because $\bar{E}[X \mid \mathcal{A}]$ is $\mathcal{A}$-measurable. Let $B \in \mathcal{B}$. Applying the Internal Approximation Theorem to the set of equivalence classes of Ω modulo $\mathcal{A}$, we see that there is an $A \in \mathcal{A}$ such that $P(A \triangle B) = 0$. Then the function $G(X) \cdot \mathbb{1}_A$ lifts $y \cdot \mathbb{1}_B$ and $X \cdot \mathbb{1}_A$ lifts $x \cdot \mathbb{1}_B$. Therefore

$$E[y \cdot \mathbb{1}_B] = \mathrm{st}(\bar{E}[G(X) \cdot \mathbb{1}_A]) = \mathrm{st}(\bar{E}[X \cdot \mathbb{1}_A]) = E[x \cdot \mathbb{1}_B].$$

This shows that $y = E[x \mid \mathcal{B}] = g(x)$, so G lifts g. □

Proposition 6.7. *Let $\mathcal{B}$ be a σ-algebra of subsets of Ω and let S be the set of all equivalence classes of $\mathcal{B}$-measurable functions $x \in \mathrm{rv}(\Omega, M)$.*

(a) *If $\mathcal{B}$ is generated by an internal algebra $\mathcal{A}$ of subsets of Ω, then S is neoclosed. In fact, $S = \mathrm{st}(A)$ where A is the internal set of all $\mathcal{A}$-measurable $X \in \mathrm{RV}(\Omega, {}^*M)$.*

(b) *If $\mathcal{B}$ is generated by the intersection of a countable decreasing chain of internal algebras $\mathcal{A}_n$ of subsets of Ω, then S is neoclosed.*

Proof. (a) First let $x \in \mathrm{st}(A)$ and let $X \in A$ lift x. Then X is $\mathcal{A}$-measurable. For each closed set $B \subseteq M$, $x^{-1}(B)$ is within a null set of $X^{-1}(\mathrm{st}^{-1}(B))$, which belongs to $\mathcal{B}$. This shows that $x \in S$, and $S \supseteq \mathrm{st}(A)$.

Now let $x \in S$. We may take x to be $\mathcal{B}$-measurable. Let h be the internal function on Ω where $h(\omega)$ is the set of all $\alpha \in \Omega$ such that ω and α belong to exactly the same sets of $\mathcal{A}$. Let Γ be the range of h. Then $h(\omega) = h(\alpha)$ implies $x(\omega) = x(\alpha)$. Let $y \in \mathrm{rv}(\Gamma, M)$ be defined by $y(h(\omega)) = x(\omega)$. Then y is Loeb measurable on Γ and therefore has lifting $Y \in \mathrm{RV}(\Gamma, {}^*M)$. Define $Z \in \mathrm{RV}(\Omega, {}^*M)$ by $Z(\omega) = Y(h(\omega))$. Then $Z \in A$ and Z lifts x. This shows that $x \in \mathrm{st}(A)$, and $S \subseteq \mathrm{st}(A)$.

(b) Let A_n be the internal set of all $\mathcal{A}_n$-measurable $X \in \mathrm{RV}(\Omega, {}^*M)$. We shall show that $S = \mathrm{st}(\bigcap_n A_n)$. First let $x \in \mathrm{st}(\bigcap_n A_n)$ and let $X \in \bigcap_n A_n$ lift x. Then X is $\mathcal{A}_n$-measurable for each $n \in \mathbb{N}$. For each closed set $B \subseteq M$, $x^{-1}(B)$ is within a null set of $X^{-1}(\mathrm{st}^{-1}(B))$, which belongs to $\mathcal{B}$. This shows that $x \in S$, and $S \supseteq \mathrm{st}(\bigcap_n A_n)$.

By part (a), $S \subseteq \mathrm{st}(A_n)$ for each n, and therefore

$$S \subseteq \bigcap_n \mathrm{st}(A_n) = \mathrm{st}\Big(\bigcap_n A_n\Big). \qquad \square$$

Given two random variables $x \in \mathrm{rv}(\Omega, M)$ and $y \in \mathrm{rv}(\Gamma, M)$ and σ-algebras $\mathcal{B}$ on Ω and $\mathcal{C}$ on Γ, we shall write $x \mid \mathcal{B} \equiv y \mid \mathcal{C}$ iff

$$\mathrm{law}(E[\psi(x) \mid \mathcal{B}]) = \mathrm{law}(E[\psi(y) \mid \mathcal{C}]) \quad \text{for all bounded continuous } \psi: M \to \mathbb{R}.$$

We shall need the following result from [20], which is a stronger form of Proposition 2.5.

Proposition 6.8. *Let $\mathscr{B}$ be a σ-algebra on Ω which is generated by an internal algebra $\mathscr{A}$ of subsets of Ω such that all partition classes of $\mathscr{A}$ are infinite and of the same internal size. Let $\mathscr{C}$ be a σ-algebra on another probability space Γ.*

(a) (Universality) *For any $y \in \mathrm{rv}(\Gamma, M)$, there exists $x \in \mathrm{rv}(\Omega, M)$ such that $x \mid \mathscr{B} \equiv y \mid \mathscr{C}$.*

(b) (Homogeneity) *If $x, y \in \mathrm{rv}(\Omega, M)$, then $x \mid \mathscr{B} \equiv y \mid \mathscr{B}$ if and only if there is an internal permutation h of Ω such that $h(\mathscr{A}) = \mathscr{A}$ and $x(h\omega) = y(\omega)$ P-almost surely.*

(c) (Saturation) *For any $x \in \mathrm{rv}(\Omega, M)$ and $\bar{x}, \bar{y} \in \mathrm{rv}(\Gamma, M)$ such that $x \mid \mathscr{B} \equiv \bar{x} \mid \mathscr{C}$, there exists $y \in \mathrm{rv}(\Omega, M)$ such that $(x, y) \mid \mathscr{B} \equiv (\bar{x}, \bar{y}) \mid \mathscr{C}$.*

We have given only a partial list of the known liftable functions. Many results called lifting theorems in the literature state in our context that certain functions are liftable. For example, the adapted lifting theorem in [20] shows that every term of adapted probability logic represents a liftable function from a space of the form $\mathrm{rv}(\Omega, \mathrm{rv}([0, 1]^c, M))$ into $\mathrm{rv}(\Omega, \mathrm{rv}([0, 1]^d, \mathbb{R})$.

7. Expected values

In the preceding section we saw that the expected value $E[x]$ and conditional expectation $E[x \mid \mathscr{B}]$ are liftable on each bounded space $\mathrm{rv}(\Omega, [a, b])$. On the space $\mathrm{rv}(\Omega, \mathbb{R})$ of all real valued random variables, the expected value is only partially defined and is not liftable. Countable conjunctions and disjunctions are needed in order to express properties of the expected value in the language $\mathscr{L}$. In this section we obtain forcing criteria for such properties. We shall need the notion of a uniformly integrable sequence, which plays a role similar to a tight sequence when dealing with expected values.

In the remainder of this paper we shall often use Chebyshev's inequality, which states that if $x \in \mathrm{rv}(\Omega, \mathbb{R})$ is integrable then for all $r \geq 0$,

$$r \cdot P[|x| \geq r] \leq E[|x|].$$

It follows that for integrable $x, y \in \mathrm{rv}(\Omega, \mathbb{R})$,

$$\hat{\rho}(x, y)^2 \leq E[|x - y|].$$

We shall use the following elementary properties of conditional expectation without explicit mention:

$E[x + y \mid \mathscr{B}] = E[x \mid \mathscr{B}] + E[y \mid \mathscr{B}]$;

$E[E[x \mid \mathscr{B}]] = E[x]$;

If $\mathscr{B} \subseteq \mathscr{C}$, then $E[E[x \mid \mathscr{B}] \mid \mathscr{C}] = E[E[x \mid \mathscr{C}] \mid \mathscr{B}] = E[x \mid \mathscr{B}]$;

If x is $\mathscr{B}$-measurable, then $E[x \cdot y \mid \mathscr{B}] = x \cdot E[y \mid \mathscr{B}]$;

If $f: \mathbb{R} \to \mathbb{R}$ is convex, then $f(E[x \mid \mathscr{B}]) \leq E[f(x) \mid \mathscr{B}]$.

We begin by expressing limits and expected values in the language $\mathscr{L}$.

Definition. Let u_m, $m \in \mathbb{N}$, and v be variables or terms with the sort space $\mathrm{rv}(\Omega, M)$. We let the expression

$$\lim_{m\to\infty} u_m = v$$

be an abbreviation for the formula

$$\bigwedge_j \bigvee_k \bigwedge_{m\geqslant k} \hat{\rho}(u_m, v) \leqslant \frac{1}{j}$$

of $\mathscr{L}$, where ρ is the metric for M and $\hat{\rho}$ is the metric for $\mathrm{rv}(\Omega, M)$.

This formula has the usual meaning when applied to random variables with sort space $\mathrm{rv}(\Omega, M)$.

Definition. Given $a \in \mathbb{R}$ and $m \in \mathbb{N}$, define the *truncation* $a \sqcap m$ by

$$a \sqcap m = \begin{cases} m & \text{if } a \geqslant m, \\ a & \text{if } |a| < m, \\ -m & \text{if } a \leqslant -m. \end{cases}$$

Given a random variable $x \in \mathrm{rv}(\Omega, \mathbb{R})$ and $m \in \mathbb{N}$, define the function $x \sqcap m$ by $(x \sqcap m)(\omega) = x(\omega) \sqcap m$. Thus $x \sqcap m$ is the function formed by truncating x above at m and below at $-m$. By Corollary 6.3 and Propositions 4.6 and 6.4, for each $m \in \mathbb{N}$, the functions $x \mapsto x \sqcap m$ and $x \mapsto E[x \sqcap m]$ are liftable.

We shall also use the notation $a \sqcap m$ and $X \sqcap m$ when $a \in {}^*\mathbb{R}$, $X \in \mathrm{RV}(\Omega, {}^*\mathbb{R})$, and $m \in {}^*\mathbb{N}$.

Definition. Given a variable u with sort space $\mathrm{rv}(\Omega, \mathbb{R})$, we let $E[|u|] < \infty$ be an abbreviation for the formula

$$\bigvee_j \bigwedge_m E[|u \sqcap m|] \leqslant j$$

of $\mathscr{L}$, and let $E[u] = a$ be an abbreviation for the formula

$$E[|u|] < \infty \wedge \left(\lim_{m\to\infty} E[u \sqcap m] = a\right)$$

of $\mathscr{L}$.

These formulas have the usual meanings when applied to random variables $x \in \mathrm{rv}(\Omega, \mathbb{R})$. That is, the formula $E[|x|] < \infty$ is true if and only if x is integrable, and $E[x] = a$ if and only if x is integrable and has expected value a.

If $\langle r_n : n \in \mathbb{N}\rangle$ is a sequence of reals and p is a condition, then $\limsup_{n \in p} r_n$ is defined in the natural way.

Proposition 7.1. *Let $\langle x \rangle$ be a tight sequence with sort space $\mathrm{rv}(\Omega, \mathbb{R})$ and let p be a condition. Then*

$$p \Vdash E[\langle |x| \rangle] < \infty$$

if and only if

$$\lim_{m\to\infty} \limsup_{n\in p} E[|x_n \sqcap m|] < \infty.$$

Proof. The following are equivalent.

$$p \Vdash E[\langle |x| \rangle] < \infty,$$

$$p \Vdash \bigvee_j \bigwedge_m E[\langle |x \sqcap m| \rangle] \leq j,$$

$$(\forall q \subseteq p)(\exists r \subseteq q) \bigvee_j r \Vdash \bigwedge_m E[\langle |x \sqcap m| \rangle] \leq j,$$

$$(\forall q \subseteq p)(\exists r \subseteq q) \bigvee_j \bigwedge_m r \Vdash E[\langle |x \sqcap m| \rangle] \leq j,$$

$$(\forall q \subseteq p)(\exists r \subseteq q) \bigvee_j \bigwedge_m \limsup_{n\in r} E[|x_n \sqcap m|] \leq j,$$

$$(\forall q \subseteq p) \bigvee_j \bigwedge_m (\exists r \subseteq q) \limsup_{n\in r} E[|x_n \sqcap m|] \leq j,$$

$$\bigvee_j \bigwedge_m \limsup_{n\in p} E[|x_n \sqcap m|] \leq j,$$

$$\lim_{m\to\infty} \limsup_{n\in p} E[|x_n \sqcap m|] < \infty. \quad \square$$

Definition. A sequence $\langle x \rangle$ of random variables with sort space $\mathrm{rv}(\Omega, \mathbb{R})$ is *uniformly integrable* iff $\lim_{m\to\infty} E[|x_n \sqcap m|]$ is finite and converges uniformly in n. Equivalently, for each $j \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $E[|x_n| \cdot \mathbb{1}_{|x_n| \geq m}] \leq 1/j$.

Definition. Let X be a lifting of $x \in \mathrm{rv}(\Omega, \mathbb{R})$. X is said to be *S-integrable* iff x is integrable and $\mathrm{st}(\bar{E}[|X|]) = E[|x|]$. Equivalently, $\bar{E}[|X|]$ is finite and $\lim_{m\to\infty} \mathrm{st}(\bar{E}[|X_n| \cdot \mathbb{1}_{|X_n| \geq m}]) = 0$. A lifting $\langle X_J : j \in {}^*\mathbb{N} \rangle$ of a sequence $\langle x_n : n \in \mathbb{N} \rangle$ is S-integrable iff X_n is S-integrable for each $n \in \mathbb{N}$.

7.2. (i) *If x is integrable, then $\{x\}$ is uniformly integrable.*

(ii) *Every integrable $x \in \mathrm{rv}(\Omega, \mathbb{R})$ has an S-integrable lifting. In fact, for every lifting X of x, $X \sqcap J$ is an S-integrable lifting of x for all sufficiently small infinite $J \in {}^*\mathbb{N}$.*

(iii) *If X is an S-integrable lifting of x, then for every internal set $A \subseteq \Omega$,* $\mathrm{st}(\bar{E}[X \cdot \mathbb{1}_A]) = E[x \cdot \mathbb{1}_A]$.

(iv) *X is S-integrable if and only if for every internal set A with $P(A) = 0$,* $\bar{E}[X \cdot \mathbb{1}_A] \approx 0$.

Part (i) is immediate. For parts (ii)–(iv) see, e.g., [28].

Lemma 7.3. *Let* $x_n \in \mathrm{rv}(\Omega, \mathbb{R})$ *for each* $n \in \mathbb{N}$.

(i) *If* $\{E[|x_n|]: n \in \mathbb{N}\}$ *is bounded, then* $\langle x_n \rangle$ *is tight.*

(ii) *If* $\{E[(x_n)^2]: n \in \mathbb{N}\}$ *is bounded, then* $\langle x_n \rangle$ *is tight and uniformly integrable.*

Proof. Let b and c be bounds in parts (i) and (ii). (i) follows from the inequality

$$m \cdot P[|x_n| \geqslant m] \leqslant E[|x_n|] \leqslant b.$$

(ii) follows from (i) and the inequalities

$$E[|x_n|] \leqslant 1 + E[(x_n)^2],$$

$$m \cdot E[|x_n| \cdot \mathbb{1}_{|x_n| \geqslant m}] \leqslant E[(x_n)^2] \leqslant c. \quad \square$$

Lemma 7.4. *Suppose* $\langle x_n : n \in \mathbb{N} \rangle$ *is a tight uniformly integrable sequence in* $\mathrm{rv}(\Omega, \mathbb{R})$ *and* $\langle X_J : j \in {}^*\mathbb{N} \rangle$ *is an S-integrable lifting. Then for all sufficiently small infinite* J, X_J *is S-integrable.*

Proof. For each m and n in $\mathbb{N}$, $\mathrm{st}(\bar{E}[|X_n|]) = E[|x_n|]$, and hence

$$\mathrm{st}(\bar{E}[|X_n| \cdot \mathbb{1}_{|X_n| \geqslant m+1}]) \leqslant E[|x_n| \cdot \mathbb{1}_{|x_n| \geqslant m}].$$

Then by ω_1-saturation, there is an infinite $K \in {}^*\mathbb{N}$ such that for each $j \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for all $n \leqslant K$,

$$\bar{E}[|X_n| \cdot \mathbb{1}_{|X_n| \geqslant m}] \leqslant 1/j.$$

We may also take K so that $\bar{E}[|X_n|]$ is finite for all $n \leqslant K$. It follows that X_n is S-integrable for all $n \leqslant K$. $\square$

Proposition 7.5. *Let* $\langle x \rangle$ *be a uniformly integrable tight sequence with sort space* $\mathrm{rv}(\Omega, \mathbb{R})$. *Then for every condition* p *and real number* a,

(i) $p \Vdash E[|\langle x \rangle|] < \infty$;

(ii) $p \Vdash E[\langle x \rangle] = a$ *if and only if* $\lim_{n \in p} E[x_n] = a$.

Proof. (i) follows from uniform integrability and Proposition 7.1.

To prove (ii), let $\langle X_J \rangle$ be an S-integrable lifting of $\langle x \rangle$, which exists by 7.2. Then $\bar{E}[X_n] \approx E[x_n]$ for all $n \in \mathbb{N}$. Therefore $\lim_{n \in p} E[x_n] = a$ if and only if $\bar{E}[X_J] \approx a$ for all sufficiently small infinite $J \in {}^*p$. By Theorem 5.9, $p \Vdash E[\langle x \rangle] = a$ if and only if $E[\mathrm{st}(X_J)] = a$ for all sufficiently small infinite $J \in {}^*p$. By Lemma 7.4, X_J is S-integrable and hence $\bar{E}[X_J] \approx E[\mathrm{st}(X_J)]$ for all sufficiently small infinite J. Therefore (ii) holds. $\square$

We now turn to conditional expectations. In the following, let $\mathscr{A}$ be an internal algebra of subsets of Ω, and let $\mathscr{B}$ be the σ-algebra generated by $\mathscr{A}$. By Proposition 6.6, for each $m \in \mathbb{N}$ the function $x \mapsto E[x \sqcap m \mid \mathscr{B}]$ is liftable.

Definition. Given variables u and v with sort space $\mathrm{rv}(\Omega, \mathbb{R})$, we let $E[u \mid \mathcal{B}] = v$ be an abbreviation for the formula

$$E[|u|] < \infty \bigwedge \Big(\lim_{m \to \infty} E[u \sqcap m \mid \mathcal{B}] = v \Big)$$

of $\mathcal{L}$.

Note that the formula $E[x \mid \mathcal{B}] = y$ of $\mathcal{L}$ is true if and only if x is integrable and has conditional expectation y with respect to $\mathcal{B}$.

Lemma 7.6. *If X is an S-integrable lifting of x, then $\bar{E}[X \mid \mathcal{A}]$ is an S-integrable lifting of $E[x \mid \mathcal{B}]$.*

Proof. By Proposition 6.6, $\bar{E}[X \sqcap m \mid \mathcal{A}]$ lifts $E[x \sqcap m \mid \mathcal{B}]$ for each $m \in \mathbb{N}$, and therefore $\bar{E}[X \mid \mathcal{A}]$ lifts $E[x \mid \mathcal{B}]$. Since $|x|$ is integrable, $E[|x| \mid \mathcal{B}]$ is integrable. Moreover,

$$\bar{E}[\bar{E}[|X| \mid \mathcal{A}]] = \bar{E}[|X|] \approx E[|x|] = E[E[|x| \mid \mathcal{B}]],$$

so $\bar{E}[X \mid \mathcal{A}]$ is S-integrable. □

Proposition 7.7. *Let $\langle x \rangle$ and $\langle y \rangle$ be tight sequences with sort space* $\mathrm{rv}(\Omega, \mathbb{R})$, *and let $\langle x \rangle$ be uniformly integrable. Then for every condition p,*

$$p \Vdash E[\langle x \rangle \mid \mathcal{B}] = \langle y \rangle \quad \textit{if and only if} \quad \lim_{n \in p} \hat{\rho}(E[x_n \mid \mathcal{B}], y_n) = 0.$$

Proof. Let $\langle X_J \rangle$ be an S-integrable lifting of $\langle x_n \rangle$, and let $\langle Y_J \rangle$ be a lifting of $\langle y_n \rangle$. By Lemma 7.6, $\langle \bar{E}[X_J \mid \mathcal{A}] \rangle$ is an S-integrable lifting of $\langle E[x_n \mid \mathcal{B}] \rangle$. Then $\bar{E}[X_n \mid \mathcal{A}] \approx E[x_n \mid \mathcal{B}]$ for all $n \in \mathbb{N}$. Therefore $\lim_{n \in p} \hat{\rho}(E[x_n \mid \mathcal{B}], y_n) = 0$ if and only if $\bar{E}[X_J \mid \mathcal{A}] \approx Y_J$ for all sufficiently small infinite $J \in {}^*p$. By Theorem 5.9, $p \Vdash E[\langle x \rangle \mid \mathcal{B}] = \langle y \rangle$ if and only if $E[\mathrm{st}(X_J) \mid \mathcal{B}] = \mathrm{st}(Y_J)$ for all sufficiently small infinite $J \in {}^*p$. By Lemmas 7.4 and 7.6, X_J is S-integrable and $\bar{E}[X_J \mid \mathcal{A}] \approx E[\mathrm{st}(X_J) \mid \mathcal{B}]$ for all sufficiently small infinite J. The required equivalence follows. □

8. Applications to stochastic processes

In this section we shall illustrate the use of our results by giving three applications to stochastic processes. With our framework in place, proofs will be elementary arguments involving standard discrete time processes. The plan will be to prove a statement by showing that it is forced by every condition and applying the main Theorem 5.3. In each case, the statement will be in the ∀∃ quantifier form. We shall sacrifice some generality in order to state the applications without too much technical background from probability theory.

In the literature, continuous time stochastic processes are usually taken to be families of random variables x_t, where t ranges over $[0, 1]$ or $[0, \infty)$. In order to illustrate our method in as simple a setting as possible, we shall present applications dealing with stochastic processes of the form x_t where t ranges over a countable dense subset $\mathbb{B}$ of $[0, \infty)$. Each of the applications given here can easily be extended to a continuous time result. We shall indicate how to do this at the end of this section. The advantage of restricting t to a countable set $\mathbb{B}$ of times is that we will be able to use the countable conjunctions and disjunctions in the language $\mathscr{L}$ to quantify over all times in $\mathbb{B}$.

Definition. For each $n \in \mathbb{N}$, let $\mathbb{B}_n$ be the set of all multiples of 2^{-n} in $[0, 2^n)$. Let $\mathbb{B}$ (the set of binary rationals) be the set $\mathbb{B} = \bigcup_n \mathbb{B}_n$. Note that $\mathbb{B}$ is dense in $[0, \infty)$. For each $t \in [0, \infty)$, let t_n be the greatest $s \in \mathbb{B}_n$ such that $s \leq t$.

By a *stochastic process* in M we shall mean an element $x \in \mathrm{rv}(\Omega, M^{\mathbb{B}})$, where $M^{\mathbb{B}}$ has the product topology. Given $c \in M^{\mathbb{B}}$, the value of c at t will be denoted by c_t. For each $t \in \mathbb{B}$, the function $c \mapsto c_t$ is continuous on $M^{\mathbb{B}}$, so the function $x \mapsto x_t$ is liftable from $\mathrm{rv}(\Omega, M^{\mathbb{B}})$ into $\mathrm{rv}(\Omega, M)$.

An element $c \in M^{\mathbb{B}}$ will be called an *n-step function* if $c_t = c_{t_n}$ for all $t \in \mathbb{B}$. A process $x \in \mathrm{rv}(\Omega, M^{\mathbb{B}})$ is called an *n-step process* if $x_t = x_{t_n}$ for all $t \in \mathbb{B}$.

The following lemma characterizes lifting and tightness in the space $M^{\mathbb{B}}$. It reduces the question of tightness in $\mathrm{rv}(\Omega, \mathbb{R}^{\mathbb{B}})$ to the question of tightness in $\mathrm{rv}(\Omega, \mathbb{R})$.

Lemma 8.1. (i) *Let $X \in \mathrm{RV}(\Omega, {}^*(M^{\mathbb{B}}))$ and $x \in \mathrm{rv}(\Omega, M^{\mathbb{B}})$. Then X lifts x if and only if X_t lifts x_t for all $t \in \mathbb{B}$.*

(ii) *$X \in RV(\Omega, {}^*(M^{\mathbb{B}}))$ is near-standard if and only if X_t is near-standard for all $t \in \mathbb{B}$.*

(iii) *A set $C \subseteq \mathrm{rv}(\Omega, M^{\mathbb{B}})$ is tight if and only if for each $t \in \mathbb{B}$ the set $C_t = \{x_t : x \in C\}$ is tight.*

Proof. Since $M^{\mathbb{B}}$ has the product topology, a point $Z \in {}^*(M^{\mathbb{B}})$ is near-standard if and only if Z_t is near-standard for all $t \in \mathbb{B}$, and if Z is near-standard then $(\mathrm{st}(Z))_t = \mathrm{st}(Z_t)$ for all $t \in \mathbb{B}$. To prove (i), we note that the following are equivalent.

X lifts x;

$P\{\omega: \mathrm{st}(X(\omega)) = x(\omega)\} = 1;$

$P\Big(\bigcap_{t\in\mathbb{B}} \{\omega: \mathrm{st}(X_t(\omega)) = x_t(\omega)\}\Big) = 1;$

for each $t \in \mathbb{B}$, $\quad P\{\omega: \mathrm{st}(X_t(\omega)) = x_t(\omega)\} = 1;$

for each $t \in \mathbb{B}$, $\quad X_t$ lifts x_t.

Part (ii) follows at once from part (i).

By Proposition 4.5, if $C \subseteq \mathrm{rv}(\Omega, M^{\mathbb{B}})$ is tight, then C_t is tight for each $t \in \mathbb{B}$. Suppose that C_t is tight for all $t \in \mathbb{B}$. By Theorem 3.1, for each $t \in \mathbb{B}$ there is an internal set $A_t \subseteq \mathrm{ns}(\Omega, {}^*M)$ such that $C_t \subseteq \mathrm{st}(A_t)$. Let

$$B = \{X : X_t \in A_t \text{ for all } t \in \mathbb{B}\}.$$

B is a Π_1^0 set. Part (ii) shows that $B \subseteq \mathrm{ns}(\Omega, {}^*(M^{\mathbb{B}}))$. We wish to show that $C \subseteq \mathrm{st}(B)$. Suppose $x \in C$. Then for each $t \in \mathbb{B}$, x_t belongs to C_t and thus has a lifting $Y_t \in A_t$. By ω_1-saturation there is an $X \in \mathrm{RV}(\Omega, {}^*(M^{\mathbb{B}}))$ such that $X_t = Y_t$ for all $t \in \mathbb{B}$. Then $X \in B$, and by part (i), X lifts x. This shows that $C \subseteq \mathrm{st}(B)$. Applying Theorem 3.1, we see that C is tight, and (iii) is proved. □

Blanket Assumption. We continue to assume that Ω is a hyperfinite set and P is the Loeb counting measure on Ω. We shall let $\langle \mathcal{G}_t : t \in \mathbb{B} \rangle$ be a family of internal algebras $\mathcal{G}_s$ of subsets of Ω such that all $\mathcal{G}_s$ partition classes have the same internal size, and whenever $s < t$, $\mathcal{G}_s$ is a proper subset of $\mathcal{G}_t$. Let $\mathcal{F}_t$ be the σ-algebra generated by $\mathcal{G}_t$.

Remark. By Proposition 6.6, for each real interval $[a, b]$ and each $t \in \mathbb{B}$, the function $x \mapsto E[x \mid \mathcal{F}_t]$ is liftable from $\mathrm{rv}(\Omega, [a, b])$ into $\mathrm{rv}(\Omega, [a, b])$. By Proposition 6.7, the set of $\mathcal{F}_t$-measurable functions is neoclosed for each $t \in \mathbb{B}$.

Definition. By an *adapted process* we mean a stochastic process $x : \Omega \to M^{\mathbb{B}}$ such that for all $t \in \mathbb{B}$, x_t is $\mathcal{F}_t$-measurable. Proposition 6.7 shows that the set of adapted processes is a countable intersection of neoclosed sets, and hence is itself neoclosed.

Definition. A *submartingale* is a stochastic process $x \in \mathrm{rv}(\Omega, \mathbb{R}^{\mathbb{B}})$ such that

(a) x is adapted,
(b) x_t is integrable for each $t \in \mathbb{B}$, and
(c) $E[x_s \mid \mathcal{F}_t] \geq x_t$ almost surely whenever $s \geq t$ in $\mathbb{B}$.

If equality holds in (c), x is called a *martingale*.

We shall call x an *n-submartingale* iff x is an n-step process such that (a)–(c) hold for $s, t \in \mathbb{B}_n$. Again, if equality holds in (c), we call x an *n-martingale*.

The notions of an adapted process, submartingale, and martingale depend on the family of σ-algebras $\mathcal{F}_t$, which will remain fixed throughout our discussion.

We mention two classical results from the literature (e.g., see [7]). First, if x is a martingale, then $|x|$ is a submartingale. Second, if x is a submartingale, then for almost all ω, the path $x_{\cdot}(\omega)$ has right and left limits at all $t \in [0, 1]$.

Theorem 8.2. *Let x and y be nonnegative submartingales in $\mathrm{rv}(\Omega, \mathbb{R}^{\mathbb{B}})$. Let z be an $\mathcal{F}_0$-measurable random variable such that for each $\omega \in \Omega$, $z(\omega)$ is equal to either $x_0(\omega)$ or $-y_0(\omega)$. Then there exists a martingale $m \in \mathrm{rv}(\Omega, \mathbb{R}^{\mathbb{B}})$ such that $m_0 = z$ and for all ω and t, $m_t(\omega)$ is equal to either $x_t(\omega)$ or $-y_t(\omega)$.*

Proof. For each $n \in \mathbb{N}$, let $x_t^n = x_{t_n}$ and $y_t^n = y_{t_n}$. For each $t \in \mathbb{B}_k$, we have $x_t^n = x_t$ and $y_t^n = y_t$ for all $n \geqslant k$. Therefore $\langle x_t^n \rangle$ and $\langle y_t^n \rangle$ are tight uniformly integrable approximations of x_t and y_t. Fix an $n \in \mathbb{N}$. x^n and y^n are n-submartingales. We shall define m_t^n with the required properties by induction on $t \in \mathbb{B}_n$. We let $m_0^n = z$. Suppose m_s^n has been defined for $s < t$ in $\mathbb{B}_n$ so that whenever $r < s < t$ in $\mathbb{B}_n$,

(1) $m_s^n(\omega) \in \{x_s^n(\omega), -y_s^n(\omega)\}$,

(2) m_s^n is $\mathcal{F}_s$-measurable,

(3) $E[m_s^n \mid \mathcal{F}_r] = m_r^n$ almost surely.

Let $s = t - 1/2^n$. We have

(4) $E[-y_t^n \mid \mathcal{F}_s] \leqslant -y_s^n \leqslant m_s^n \leqslant x_s^n \leqslant \bar{E}[x_t^n \mid \mathcal{F}_s]$.

We wish to apportion the values of $m_t^n(\omega)$ among $x_t^n(\omega)$ and $-y_t^n(\omega)$ in such a way that m^n is an n-martingale up to time t. To do this, choose an $\mathcal{F}_t$-measurable random variable $\alpha \in \mathrm{rv}(\Omega, [0, 1])$ such that for each $b \in [0, 1]$,

$$P[\alpha \leqslant b \mid \mathcal{F}_s] = b.$$

Such an α exists by Proposition 6.8(c) (saturation) applied to the space $\Omega/\mathcal{G}_t$ of $\mathcal{G}_t$ partition classes and the algebra $\mathcal{G}_s/\mathcal{G}_t$. It follows from (4) that there is an $\mathcal{F}_s$-measurable random variable $\beta \in \mathrm{rv}(\Omega, [0, 1])$ such that for almost all ω,

$$E[(x_t^n \cdot \mathbb{I}_{\alpha \geqslant \beta}) - (y_t^n \cdot \mathbb{I}_{\alpha < \beta}) \mid \mathcal{F}_s](\omega) = m_s^n(\omega).$$

Therefore

$$m_t^n = (x_t^n \cdot \mathbb{I}_{\alpha \geqslant \beta}) - (y_t^n \cdot \mathbb{I}_{\alpha < \beta})$$

satisfies properties (1)–(3) for t. This completes the induction.

For each $t \in \mathbb{B}$, let $m_t^n = m_{t_n}^n$. Then m^n is an n-martingale, and $m_t^n(\omega) \in \{x_t^n(\omega), -y_t^n(\omega)\}$ for all $t \in \mathbb{B}$ and all ω. Since $\langle x_t^n \rangle$ and $\langle y_t^n \rangle$ are tight and uniformly integrable, $\langle m_t^n \rangle$ is tight and uniformly integrable. Let $s \leqslant t$ in $\mathbb{B}$ and let p be a condition. By Proposition 7.7,

$$p \Vdash E[\langle m_t^n \rangle \mid \mathcal{F}_s] = \langle m_s^n \rangle.$$

The set

$$\{(x, y, m) \colon m_t(\omega) \in \{x_t(\omega), -y_t(\omega)\}\ P\text{-almost surely}\}$$

is neoclosed, and hence

$$p \Vdash \langle m_t^n(\omega) \rangle \in \{\langle x_t^n(\omega) \rangle, -\langle y_t^n(\omega) \rangle\}\ P\text{-almost surely}.$$

Therefore p forces the sentence

$$(\exists v)\Big[v \text{ is a martingale and}$$

$$\bigwedge_{t\in\mathbb{B}} [v_t(\omega) \in \{\langle x_t^n(\omega)\rangle, -\langle y_t^n(\omega)\rangle\} \ P\text{-almost surely}]\Big].$$

The result now follows from Theorem 5.3. $\square$

Remark. With somewhat more work, the above result can be proved for local submartingales. Theorems along this line for the special case $x = y$ were obtained by Gilat [11], Barlow [4], and Perkins [26].

Let us call a process $y \in \mathrm{rv}(\Omega, \mathbb{R}^{\mathbb{B}})$ *increasing* iff whenever $s < t$ in $\mathbb{B}$, $y_s(\omega) \leq y_t(\omega)$ P-almost surely. The following finite difference notation will be useful in the next two applications.

Definition. Given an $x \in \mathrm{rv}(\Omega, \mathbb{R}^{\mathbb{B}})$, $n \in \mathbb{N}$, and $t \in \mathbb{B}_n$, let

$$\Delta_n x_t = x_u - x_t \quad \text{where } u = t + 1/2^n$$

and

$$\sum_{s<t}^{n} x_s = \sum \{x_s : s < t \text{ and } s \in \mathbb{B}_n\}.$$

Theorem 8.3. *Let x be a submartingale in* $\mathrm{rv}(\Omega, \mathbb{R}^{\mathbb{B}})$. *Suppose that for each $t \in \mathbb{B}$, the expected values*

$$E\bigg[\Big(\sum_{s<t}^{n} E[\Delta_n x_s \mid \mathcal{F}_s]\Big)^2\bigg]$$

exist and are bounded uniformly in n. Then there exists a martingale m and an increasing process z with values in $\mathbb{N} \cup \{0\}$ such that $m + z = x$.

Remark. The Doob–Meyer decomposition theorem (e.g., see [9]) states that for every submartingale x which is bounded below there exists a martingale m and an increasing 'predictable' process y such that $y_0 = 0$ and $m + y = x$. Theorem 8.3 is an analogous result but with z having values in $\mathbb{N}$ instead of being predictable. Note that since z is increasing with values in $\mathbb{N} \cup \{0\}$, its paths are step functions with positive jumps of integer size.

Proof. Let a^n be the increasing n-step process

$$a_0^n = 0, \qquad a_t^n = \sum_{s<t}^{n} E[\Delta_n x_s \mid \mathcal{F}_s].$$

Then $\Delta_n a_t^n = E[\Delta_n x_t \mid \mathcal{F}_t]$ is $\mathcal{F}_t$-measurable, and $E[a_t^n] = E[x_t]$. The hypothesis states that for each $t \in \mathbb{B}$, the values $E[(a_t^n)^2]$ exist and are bounded uniformly in

n. Fix n. Let $t \in \mathbb{B}_n$. By Proposition 6.8(c) (saturation), for each $t \in \mathbb{B}_n$ we may choose an $\mathcal{F}_{t+1/2^n}$-measurable random variable $b_t \in \mathrm{rv}(\Omega, \mathbb{N})$ which has the Poisson distribution with $\Delta_n a_t^n$ given $\mathcal{F}_t$, that is,

$$P[b_t = k \mid \mathcal{F}_t] = \mathrm{e}^{-\lambda} \cdot \lambda^k / k! \quad \text{where } \lambda = \Delta_n a_t^n.$$

Moreover, b_t may be chosen so that for all $s > t$ in $\mathbb{B}_n$,

$$\text{(1)} \qquad E[b_t \cdot \Delta_n x_s \mid \mathcal{F}_t] = E[b_t \mid \mathcal{F}_t] \cdot E[\Delta_n x_s \mid \mathcal{F}_t].$$

Define an n-step process z_t^n by

$$z_0^n = 0, \qquad \Delta_n z_t^n = b_t.$$

Then z^n is increasing and z_t^n is $\mathcal{F}_t$-measurable and has values in $\mathbb{N} \cup \{0\}$. Let $x_t^n = x_{t_n}$ and let $m_t^n = x_t^n - {}_t^n$. Then m_t^n is $\mathcal{F}_t$-measurable. Since $E[b_t \mid \mathcal{F}_t] = \Delta_n a_t^n$,

$$\text{(2)} \qquad E[m_t^n \mid \mathcal{F}_s] = m_t^n \text{ almost surely} \quad \text{whenever } s < t \text{ in } \mathbb{B}_n.$$

As in the preceding proof, for each $t \in \mathbb{B}$, $\langle x_t^n \rangle$ is a tight uniformly integrable sequence which approximates x_t. To show that $\langle z_t^n \rangle$ and $\langle m_t^n \rangle$ are tight and uniformly integrable, we shall compute second moments and use Lemma 7.3. The formulas for mean and second moment of a Poisson distribution give

$$E[\Delta_n z_t^n \mid \mathcal{F}_t] = \Delta_n a_t^n, \qquad E[(\Delta_n z_t^n)^2 \mid \mathcal{F}_t] = (\Delta_n a_t^n)^2 + \Delta_n a_t^n.$$

Using (1), a routine computation shows that for each $t \in \mathbb{B}_n$

$$E[(z_t^n)^2] = E[(a_t^n)^2] + E[a_t^n].$$

It follows that the values $E[(z_t^n)^2]$ are bounded uniformly in n. Therefore by Lemma 7.3, the sequence $\langle z^n \rangle$ is tight and uniformly integrable. Since $\langle x_t^n \rangle$ is tight and uniformly integrable, so is the difference $\langle m_t^n \rangle = \langle x_t^n - z_t^n \rangle$.

As in the preceding example, Proposition 7.7 and (2) show that any condition forces the statement that $\langle m_t^n \rangle$ is a martingale. Thus any condition forces that there exists a martingale m and an increasing process z with values in $\mathbb{N} \cup \{0\}$ such that $m + z = \langle x \rangle$. The result now follows by Theorem 5.3. □

Our third application will be an analogue of the Peano existence theorem for stochastic differential equations. The Peano existence theorem states that for every bounded continuous real function $f(t, x)$ and real initial value x_0, there exists a real function $x : [0, \infty) \to \mathbb{R}$ such that

$$x(0) = x_0 \quad \text{and} \quad x(t) = \int_0^t f(s, x(s))\, \mathrm{d}s.$$

This theorem has a simple nonstandard proof which is often used to motivate the method of liftings (see, for example, [22]). It can also be proved by forcing using the easy deterministic case of Theorem 5.3. We leave this as an exercise for the reader. Instead we apply Theorem 5.3 to prove an existence theorem for stochastic integral equations with respect to a martingale.

Example 8.4 (An existence theorem for stochastic integral equations). *Let m be a martingale such that for some constant c,*

$$E[(\Delta_n m_t)^2 \mid \mathscr{F}_t] \leqslant c/2^n$$

for all $n \in \mathbb{N}$ and $t \in \mathbb{B}_n$. Let f be a bounded uniformly continuous function from $\mathbb{R}$ into $\mathbb{R}$. Then there is a martingale x such that $x = 0$ and for each $t \in \mathbb{B}$,

$$x_t = \lim_{i \to \infty} \sum_{s<t}^{i} f(x)\Delta_i m_s.$$

Proof. For $n \in \mathbb{N}$ and $t \in \mathbb{B}_n$ let x_t^n be the solution of the difference equation

$$x_0^n = 0, \qquad x_t^n = \sum_{s<t}^{n} f(x_s^n)\Delta_n m_s.$$

Extend x_t^n to an n-step function by putting $x_t^n = x_{t_n}^n$. Then for each $t \in \mathbb{B}$ x_t^n is $\mathscr{F}_{t_n}$-measurable.

Let b be a uniform bound for $|f|$. Then for each $t \in \mathbb{B}_n$,

$$E[(x_t^n)^2] = E\left[\sum_{s<t}^{n} (f(x_s^n)\Delta_n m_s)^2\right] \leqslant b^2 \cdot \sum_{s<t}^{2} E[(\Delta_n m_s)^2] \leqslant b^2 \cdot c \cdot t.$$

Thus by Lemma 7.3, the sequence $\langle x_t^n \rangle$ is tight and uniformly integrable. Therefore each x_t^n is integrable. Since $f(x_t^n)$ is bounded and adapted and m_t is a martingale, it follows that x_t^n is an n-martingale. By Proposition 7.7, for any condition p and any $s \leqslant t$ in $\mathbb{B}$ we have

$$p \Vdash \langle x_0^n \rangle = 0 \quad \text{and} \quad p \Vdash E[\langle x_t^n \rangle \mid \mathscr{F}_s] = \langle x_s^n \rangle.$$

It remains to prove that for each $t \in \mathbb{B}$,

$$\text{(1)} \qquad p \Vdash \langle x_t^n \rangle = \lim_{i \to \infty} \langle y_t^{n,i} \rangle,$$

where

$$y_t^{n,i} = \sum_{s<t}^{i} f(x_s^n)\Delta_i m_s.$$

To prove (1) it suffices to prove

$$\text{(2)} \qquad \bigwedge_j \bigvee_k \bigwedge_{i \geqslant k} p \Vdash \hat{\rho}(\langle x_t^n \rangle, \langle y_t^{n,i} \rangle) \leqslant \frac{1}{j}.$$

To prove (2) for all p it suffices to prove

$$\text{(3)} \qquad \bigwedge_j \bigvee_k \bigwedge_{i \geqslant k} \bigwedge_{n \geqslant i} \hat{\rho}(x_t^n, y_t^{n,i}) \leqslant \frac{1}{j}.$$

Since $\hat{\rho}(u, v)^2 \leqslant E[|u - v|] \leqslant (E[(u - v)^2])^{1/2}$, for (3) it suffices to prove

$$\text{(4)} \qquad \bigwedge_j \bigvee_k \bigwedge_{i \geqslant k} \bigwedge_{n \geqslant i} E[(x_t^n - y_t^{n,i})^2] \leqslant \frac{1}{j}.$$

Let $n \geqslant i$ and $t \in \mathbb{B}_i$. Then

$$(5) \qquad x_t^n - x_t^{n,i} = \sum_{s<t}^n (f(x_s^n) - f(x_{s_i}^n))\Delta_n m_s.$$

We wish to show that $x_s^n - x_{s_i}^n$ is small, hence $f(x_s^n) - f(x_{s_i}^n)$ is small and $x_t^n - y_t^{n,i}$ is small. Again, we shall do this by getting bounds for expected values of squares. For $u < s \in \mathbb{B}_n$ we have

$$x_s^n - x_u^n = \sum_{u \leqslant r < s}^n f(x_r^n)\Delta_n m_r,$$

and as before,

$$E[(x_s^n - x_u^n)^2] \leqslant b^2 \cdot c \cdot (s-u).$$

Then for each $\delta > 0$,

$$P[(x_s^n - x_u^n)^2 \geqslant \delta] \leqslant b^2 \cdot c \cdot (s-u)/\delta.$$

Consider any $\varepsilon > 0$. Since f is uniformly continuous, for all sufficiently large $i \in \mathbb{N}$, we have

$$P[(f(x_s^n) - f(x_u^n))^2 \geqslant \varepsilon] \leqslant \varepsilon$$

whenever $n \geqslant i$, $u < s \in \mathbb{B}_n$, and $s - u \leqslant 1/2^i$. Since f is bounded by b,

$$E[(f(x_s^n) - f(x_u^n))^2] \leqslant \varepsilon(1 + 4b^2).$$

Using (5) and the fact that $f(x_s^n) - f(x_{s_i}^n)$ is $\mathscr{F}_s$-measurable, we have

$$\begin{aligned}(6) \qquad E[(x_t^n - y_t^{n,i})^2] &= \sum_{s<t}^n E[(f(x_s^n) - f(x_{s_i}^n))^2(\Delta_n m_s)^2] \\ &= \sum_{s<t}^n E[(f(x_s^n) - f(x_{s_i}^n))^2 \cdot E[(\Delta_n m_s)^2 \mid \mathscr{F}_s]] \\ &\leqslant \varepsilon \cdot (1 + 4b^2) \cdot c \cdot t.\end{aligned}$$

We can now verify the required statement (4). Given $j \in \mathbb{N}$, take ε such that $\varepsilon \cdot (1 + 4b^2) \cdot c \cdot t \leqslant 1/j$, and choose $k \in \mathbb{N}$ such that (6) holds whenever $n \geqslant i \geqslant k$. □

Remark. With the appropriate definitions, Example 8.4 shows that under the given hypotheses, the stochastic integral equation

$$x_t = \int_0^t f(x_s)\,\mathrm{d}m_s$$

has a strong solution x_t in a hyperfinite probability space. The case of the theorem when m is a Brownian motion is in [18, Theorem 5.2], and is a strong form of a theorem of Skorokhod [29]. The theorem proved here fails if Ω is not assumed to be a hyperfinite probability space; for a counterexample see Barlow [5]. A much more general existence theorem was proved by Hoover and Perkins [16] and

Jacod and Memin [17]. This generalization can also be proved by the present method.

The preceding three applications can be extended to continuous time stochastic processes in the following way. For each $t \in [0, \infty)$ let

$$\mathcal{H}_t = \bigcap_{t<s\in\mathbb{B}} \mathcal{F}_s.$$

The following two facts are easy to check.

8.5. *If* z_t, $t \in [0, \infty)$, *is an* $\mathcal{H}_\bullet$*-submartingale with right continuous paths, then* $x_s = E[z_s \mid \mathcal{F}_s]$, $s \in \mathbb{B}$, *is an* $\mathcal{F}_\bullet$*-submartingale, and*

$$z_t = \lim_{s\downarrow t} x_s.$$

8.6. *If* x_s, $s \in \mathbb{B}$, *is an* $\mathcal{F}_\bullet$*-submartingale, then*

$$z_t = \lim_{s\downarrow t} x_s, \quad t \in [0, \infty),$$

is an $\mathcal{H}_\bullet$*-submartingale with right continuous paths.*

Using these facts, Theorems 8.2, 8.3, and 8.4 for $\mathcal{F}_\bullet$-submartingales with times in $\mathbb{B}$ imply the same theorems for right continuous $\mathcal{H}_\bullet$-submartingales with times in $[0, \infty)$. In each case, first use 8.5, then apply the theorem for $\mathcal{F}_\bullet$, and then use 8.6 to get the result for $\mathcal{H}_\bullet$.

It is not clear at this point whether Robinson's infinitesimal analysis is essential to our main results. In the present treatment, the central notions of a hyperfinite probability space, a liftable function, and the language $\mathcal{L}$ are defined in terms of Robinson's infinitesimals. However, in the deterministic case the liftable functions are just the continuous functions and Theorem 5.3 reduces to a result stated in standard terms. Moreover, once the language $\mathcal{L}$ is set up, the definition of forcing and the applications are stated purely in standard terms. It is possible to formulate a standard consequence of Theorem 5.3 which is sufficient for the applications, as follows. Choose a particular set S of liftable functions which is rich enough for the purpose at hand. "There exists a probability space (Ω, P) and a nontrivial family of σ-algebras $\mathcal{F}_t$ increasing in t such that for any sentence ϕ built up from equations over S using quantifiers and countable connectives, and any condition p, p forces ϕ if and only if ϕ is true for $(\Omega, P, \mathcal{F}_t)$." The papers of Fajardo [10] and Hoover [14] show that in many cases lifting theorems in hyperfinite adapted probability spaces can be replaced by results about well-behaved extensions of standard adapted probability spaces. Perhaps a similar approach can be used in the context of this paper.

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